Michigan Math. J. 60 (2011), 93–111

Properties of Meromorphic φ-normal Functions

Rauno Aulaskari & Jouni Rättyä

1. Introduction

Let $\mathcal{M}(\mathbb{D})$ denote the set of all meromorphic functions in the unit disc $\mathbb{D} := \{z : |z| < 1\}$ of the complex plane $\mathbb{C}$, and let $\mathcal{T}$ stand for the set of all conformal self maps of $\mathbb{D}$. The class $\mathcal{N}$ of *normal functions* consists of those $f \in \mathcal{M}(\mathbb{D})$ for which the family $\{f \circ \tau : \tau \in \mathcal{T}\}$ is normal in $\mathbb{D}$ in the sense of Montel (i.e., ∞ is a permitted limit). By Marty's theorem, $f \in \mathcal{N}$ if and only if $\sup_{\tau\in\mathcal{T}}(f \circ \tau)^{\#}(z)$ is bounded on each compact subset of $\mathbb{D}$. Moreover, Lehto and Virtanen [27] showed that $f \in \mathcal{M}(\mathbb{D})$ is normal if and only if its spherical derivative $f^{\#}(z) := |f'(z)|/(1 + |f(z)|^2)$ satisfies $\sup_{z\in\mathbb{D}} f^{\#}(z)(1 - |z|^2) < \infty$.

There is a substantial body of literature on normal functions. Apart from the cited paper by Lehto and Virtanen [27], we mention the earlier work by Noshiro [30], the survey paper by Cambell and Wickes [9], and the papers by Anderson, Clunie, and Pommerenke [1], Lohwater and Pommerenke [28], and Zalcman [41] as well as the series of papers by Gavrilov [17; 18; 19], Lappan [23; 24; 25; 26], and Yamashita [38; 39; 40]. For more recent developments, see [5; 7; 11; 13; 20] and the references therein.

The purpose of this paper is to study subsets of $\mathcal{M}(\mathbb{D})$ that are defined by the condition $f^{\#}(z) = \mathcal{O}(\varphi(|z|))$, as $|z| \to 1^-$, where the function $\varphi(r)$ admits a sufficient regularity near 1 and exceeds $1/(1-r^2)$ in growth. These sets are larger than the class $\mathcal{N}$ of normal functions, and their members will be called φ-normal functions. These concepts are made precise in Definition 1. After that we give several examples of admissible functions φ. At the end of this section we illustrate what it means to change the growth restriction of spherical derivatives from $1/(1 - |z|^2)$ of normal functions to $\varphi(|z|)$ of φ-normal functions. Statements of the main results and their connections to existing literature are given in Section 2. Proofs are presented in Sections 3–9.

Definition 1. An increasing function $\varphi\colon [0, 1) \to (0, \infty)$ is called *smoothly increasing* if

$$\varphi(r)(1 - r) \to \infty \quad \text{as } r \to 1^- \tag{1.1}$$

and

Received May 6, 2009. Revision received February 9, 2010.
This research was supported in part by MEC-Spain MTM2005-07347, MTM2007-30904-E, MTM2008-05891; ESF Research Networking Programme HCAA; and the Academy of Finland 121281.