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On the Torelli Problem and Jacobian Nullwerte in Genus Three

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1. Introduction

It is known that the set of bitangent lines of a non-hyperelliptic genus-three curve determines completely the curve, since it admits a unique symplectic structure [CaS; L]. Given this structure, one can recover an equation for the curve following a method of Riemann ([Rie]; see also [R1]): one takes an Aronhold system of bitangent lines, determines some parameters by means of some linear systems, and then writes down a *Riemann model* of the curve. Unfortunately, the parameters involved in this construction are not defined in general over the field of definition of the curve but rather over the field of definition of the bitangent lines. This is inconvenient for arithmetical applications concerned with rationality questions (cf. [O], for instance).

We propose an alternative construction, giving a model of the curve directly from a certain set of bitangent lines; this model is already defined over the field of definition of the curve. In the particular case of complex curves, our construction provides a closed solution for the non-hyperelliptic Torelli problem on genus three as follows.

Theorem 1.1. *Let $\mathcal{C}$ be a non-hyperelliptic genus-three curve defined over a field $K \subset \mathbb{C}$, and let $\omega_1, \omega_2, \omega_3$ be a K-basis of $H^0(\mathcal{C}, \Omega^1_{/K})$ and $\gamma_1, \dots, \gamma_6$ a symplectic basis of $H_1(\mathcal{C}, \mathbb{Z})$. We denote by $\Omega = (\Omega_1|\Omega_2) = \left(\int_{\gamma_j} \omega_k\right)_{j,k}$ the period matrix of $\mathcal{C}$ with respect to these bases and by $Z = \Omega_1^{-1}.\Omega_2$ the normalized period matrix. A model of $\mathcal{C}$ defined (up to normalization) over K is*

$$\left(\frac{[w_7w_2w_3][w_7w_2'w_3']}{[w_1w_2w_3][w_1'w_2'w_3']}X_1Y_1 + \frac{[w_1w_7w_3][w_1'w_7w_3']}{[w_1w_2w_3][w_1'w_2'w_3']}X_2Y_2 - \frac{[w_1w_2w_7][w_1'w_2'w_7]}{[w_1w_2w_3][w_1'w_2'w_3']}X_3Y_3\right)^2 - 4\frac{[w_7w_2w_3][w_7w_2'w_3']}{[w_1w_2w_3][w_1'w_2'w_3']}\,\frac{[w_1w_7w_3][w_1'w_7w_3']}{[w_1w_2w_3][w_1'w_2'w_3']}X_1Y_1X_2Y_2 = 0,$$

where

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