Michigan Math. J. 59 (2010), 667–678

Sharpness of the Assumptions for the Regularity of a Homeomorphism

Stanislav Hencl

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that a mapping $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ has finite (outer) distortion if $J_f(x) \geq 0$ almost everywhere and $J_f(x) = 0$ implies $|Df(x)| = 0$ a.e. Moreover, we say that a mapping $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ has finite inner distortion if $J_f(x) \geq 0$ almost everywhere and $J_f(x) = 0$ implies $|\text{adj}\, Df| = 0$ a.e. (for basic properties, examples, and applications, see e.g. [10]). Here adj A means the adjugate matrix; see Section 2 for the definition.

Our aim is to show the sharpness of the following recent result from [1] (see also [6; 7; 8; 11; 14]).

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f \in W^{1,n-1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ be a homeomorphism of finite inner distortion. Then $f^{-1} \in W^{1,1}_{\text{loc}}(f(\Omega), \mathbb{R}^n)$ and f^{-1} is a mapping of finite outer distortion. Moreover,*

$$\int_{f(\Omega)} |Df^{-1}(y)|\, dy = \int_{\Omega} |\text{adj}\, Df(x)|\, dx. \tag{1.1}$$

This statement is actually claimed in [1] only for mappings of finite outer distortion. However, with a very slight modification of the arguments given there (see Section 3 for details) it is possible to show the statement also for a wider class of mappings of finite inner distortion (see also [4]). Also formula (1.1) is not shown there, but it was previously shown under stronger assumptions in [7] and under a $W^{1,n-1}$ regularity assumption in [16]. Let us also note that the assumption that f has finite inner distortion is not artificial, because it was shown in [9, Thm. 4] that each homeomorhism such that $f \in W^{1,1}_{\text{loc}}$, $J_f \geq 0$, a.e. and $f^{-1} \in W^{1,1}_{\text{loc}}$ is necessarily a mapping of finite inner distortion.

Our aim is to show that the assumptions of Theorem 1.1 are sharp in the sense that the crucial regularity condition $|Df| \in L^{n-1}_{\text{loc}}$ cannot be weakened. From the equality (1.1) one may be tempted to believe that to conclude $Df^{-1} \in L^1$ it could be enough to assume that adj $Df \in L^1$. We show that this is not true.

Example 1.2. *Let $0 < \varepsilon < 1$ and $n \geq 3$. There exist a domain $\Omega \subset \mathbb{R}^n$ and a homeomorphism $f \in W^{1,n-1-\varepsilon}(\Omega, \mathbb{R}^n)$ such that $|\text{adj}\, Df| \in L^1(\Omega)$ and a pointwise derivative ∇f^{-1} exists a.e. in $f(\Omega)$ but $|\nabla f^{-1}| \notin L^1(f(\Omega))$.*

Received March 24, 2009. Revision received October 22, 2009.
The author was supported by MSM 0021620839.