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The Schwarz–Pick Lemma of High Order in Several Variables

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1. Introduction

Let $\mathbb{B}_n$ be the unit ball in the complex space $\mathbb{C}^n$ of dimension n. The unit disk in the complex plane is denoted by $\mathbb{D}$. For $z = (z_1, \dots, z_n)$ and $z' = (z_1', \dots, z_n')$ in $\mathbb{C}^n$, denote $\langle z, z' \rangle = z_1 \bar{z}_1' + \cdots + z_n \bar{z}_n'$ and $|z| = \langle z, z \rangle^{1/2}$.

A multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ of dimension n consists of n nonnegative integers α_j, $1 \le j \le n$; the degree of a multi-index α is the sum $|\alpha| = \sum_{j=1}^n \alpha_j$; and we denote $\alpha! = \alpha_1! \cdots \alpha_n!$. For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, let $z^\alpha = \prod_{j=1}^n z_j^{\alpha_j}$. A holomorphic function f on $\mathbb{B}_n$ can be expressed by $f(z) = \sum_\alpha a_\alpha z^\alpha$. For two multi-indexes $\alpha = (\alpha_1, \dots, \alpha_n)$ and $v = (v_1, \dots, v_n)$, let $v^\alpha = v_1^{\alpha_1}, \dots, v_n^{\alpha_n}$. Note that $v_j^{\alpha_j} = 1$ if $v_j = \alpha_j = 0$. Let $\Omega_{n,m}$ be the class of all holomorphic mappings f from $\mathbb{B}_n$ into $\mathbb{B}_m$. For $f \in \Omega_{n,m}$, if $f = (f_1, \dots, f_m)$ and $f_j(z) = \sum_\alpha a_{j,\alpha} z^\alpha$ for $j = 1, \dots, m$, we denote $f(z) = \sum_\alpha a_\alpha z^\alpha$, where $a_\alpha = (a_{1,\alpha}, \dots, a_{m,\alpha})$.

For $f \in \Omega_{1,1}$, the classical Schwarz–Pick lemma says that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2}$$

holds for $z \in \mathbb{D}$. Recently, this inequality has been generalized to the derivatives of arbitrary order by some authors [DP; MSZ; Zh]. The best result was proved in [DP]. It was proved that

$$\frac{|f^{(k)}(z)|}{1 - |f(z)|^2} \le (1 + |z|)^{k-1} \cdot \frac{k!}{(1 - |z|^2)^k} \tag{1.1}$$

holds for $f \in \Omega_{1,1}$ with $k \ge 1$ and $z \in \mathbb{D}$. The equality in (1.1) may be attained if $z = 0$, and the equality statement has been established. If $k > 1$ and $z \ne 0$, then (1.1) is a strict inequality.

Chen and Liu [CL] generalized (1.1) by proving the following Schwarz–Pick estimate for partial derivatives of arbitrary order of a function $f \in \Omega_{n,1}$:

$$\left| \frac{\partial^{|v|} f(z)}{\partial z_1^{v_1} \cdots \partial z_n^{v_n}} \right| \le n^{|v|/2} |v|! \binom{n + |v| - 1}{n - 1}^{n+2} \frac{1 - |f(z)|^2}{(1 - |z|^2)^{|v|}} (1 + |z|)^{|v|-1} \tag{1.2}$$

holds for any $z \in \mathbb{B}_n$ and multi-index $v = (v_1, \dots, v_n) \ne 0$.

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