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Local Polynomial Convexity of Certain Graphs in C^2

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1. Introduction

Let K be a compact subset of $\mathbf{C}^n$, and denote by $\hat{K}$ the polynomial convex hull of K:

$$\hat{K} = \{z \in \mathbf{C}^n : |p(z)| \le \|p\|_K \text{ for every polynomial } p \text{ in } \mathbf{C}^n\}.$$

We say that K is *polynomially convex* if $\hat{K} = K$. A compact K is called *locally polynomially convex* at $a \in K$ if there exists a closed ball $\bar{B}(a,r)$, centered at a and with radius $r > 0$, such that $\bar{B}(a,r) \cap K$ is polynomially convex. A compact $K \subset \mathbf{C}$ is polynomially convex if $\mathbf{C} \setminus K$ is connected. In higher dimensions there is no such topological characterization of polynomially convex sets, and it is usually hard to determine whether a given compact set is polynomially convex. By a well-known result of Wermer ([We]; see also [AWe, Thm. 17.1]), every totally real manifold is locally polynomially convex. Recall that a $\mathcal{C}^1$ smooth real manifold M is called *totally real* at $p \in M$ if the real tangent space $T_p M$ contains no complex line. In this paper, we are concerned with local polynomial convexity at the origin of the graph Γ_f of a $\mathcal{C}^2$ smooth function f near $0 \in \mathbf{C}$ such that $f(0) = 0$.

By the theorem of Wermer just cited, we know that if $\frac{\partial f}{\partial \bar{z}}(0) \ne 0$ then Γ_f is locally polynomially convex at the origin. Thus it remains to consider the case where $\frac{\partial f}{\partial \bar{z}}(0) = 0$. The work associated with this direction of research is too numerous to list here; instead, the reader is referred to [B1; B2; Wi] and the references given therein. The general scheme of studying local polynomially convexity of Γ_f is by pulling back Γ_f under a proper holomorphic map. Then the inverse of Γ_f is a finite union $X_1 \cup \dots \cup X_k$ of totally real graphs. Using Wermer's theorem, we conclude that the inverse of Γ_f is a finite union of *locally* polynomially convex compact sets meeting only at the origin.

Next, under some appropriate assumptions on f, one can show that there is a polynomial map $p\colon \mathbf{C}^2 \to \mathbf{C}$ such that the sets $p(X_k)$ are contained in disjoint open sectors with vertex at the origin. Then a lemma of Kallin ([Ka]; see also [Pa]) implies that $X_1 \cup \dots \cup X_k$ is locally polynomially convex at the origin. Finally, since polynomial convexity behaves nicely under proper holomorphic transformations, we conclude that Γ_f is locally polynomially convex at the origin.

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