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Dirac Operators with Periodic δ-Interactions: Spectral Gaps and Inhomogeneous Diophantine Approximation

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1. Introduction and Summary

Let $\kappa \in (0, 2\pi)$, $\Gamma = \{0, \kappa\} + 2\pi\mathbf{Z}$, $m \geq 0$, and $\beta \in \mathbf{R} \setminus \{0\}$. Let σ_1 and σ_3 stand for the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We are concerned with the spectrum of the Dirac operator H in $(L^2(\mathbf{R}))^2$, which is defined as

$$(Hf)(x) = -i\sigma_1 \frac{d}{dx} f(x) + m\sigma_3 f(x), \quad x \in \mathbf{R} \setminus \Gamma;$$

$$\begin{aligned}\operatorname{Dom}(H) = \Big\{ \big(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\big) \;\Big|\; & f_1 \in H^1(\mathbf{R}),\ f_2 \in H^1(\mathbf{R} \setminus \Gamma), \\ & f_2(x+0) - f_2(x-0) = -i\beta f_1(x) \text{ for } x \in \Gamma \Big\}.\end{aligned}$$

The operator H is self-adjoint, and the spectrum of H has the band structure. The purpose of this paper is to establish a relationship between the asymptotic behavior of the spectral gaps of H and the number-theoretical properties of parameters involved in H.

In order to formulate our main result, we describe basic spectral properties of the operator H. Toward this end, we first introduce the discriminant of H, which plays the most fundamental role in the analysis of the spectrum of H (cf. [8; 11; 14; 16, Sec. XIII]). For a parameter $\lambda \in \mathbf{R}$, let $M(\lambda, x) \in M_2(\mathbf{C})$ stand for the solution to the equations

$$\begin{cases} \big(-i\sigma_1 \frac{d}{dx} + m\sigma_3\big) Y(x) = \lambda Y(x), & x \in \mathbf{R} \setminus \Gamma, \\ Y(x+0) = \big(\begin{smallmatrix} 1 & 0 \\ -i\beta & 1 \end{smallmatrix}\big) Y(x-0), & x \in \Gamma, \end{cases}$$

subject to the initial condition

$$Y(+0) = I,$$

where I is the 2×2 identity matrix. We call $M(\lambda, x)$ the *monodromy matrix* of H. The discriminant of H is defined as

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