Primitive Permutation Groups and Their Section-Regular Partitions

PETER M NEUMANN

Dedicated to the memory of Donald G. Higman

1. Introduction

The study reported here arises out of a question asked by João Araújo (Lisbon) in an e-mail message of 19 October 2006. Let X be a finite set and let G be a group of permutations of X, that is, a subgroup of $\mathrm{Sym}(X)$. A partition of X as a disjoint union of non-empty pairwise disjoint subsets corresponds to an equivalence relation, and we shall move freely between the two concepts, using ρ to stand either for the relation or for the partition, as context demands. A section (or transversal) of ρ is then a subset S of X that contains precisely one element from each class of ρ. Given a relation ρ and one of its sections S, the two conditions

$$S^g \text{ is a section of } \rho \text{ for all } g \in G, \qquad S \text{ is a section of } \rho^g \text{ for all } g \in G$$

are of course equivalent. A relation ρ for which there exists a section S such that this condition is satisfied will be called *section-regular* relative to G or sometimes *G-regular*. In this language Araújo's question is

> *is it true that if G is primitive on X then there are no non-trivial proper G-regular partitions of X?*

The short answer is no. As usual, however, much lies behind this monosyllable.

The context of the question is this. Call the group G *synchronizing* if $G \neq \{1\}$ and there are no non-trivial proper G-regular partitions of X. Although formally different, this is in effect the same as a definition made by Araújo in his work on semigroups and automata (see Section 6 of this paper). Clearly, if G is intransitive then the partition of X into orbits is section-regular relative to G. Similarly, if G is transitive but imprimitive and ρ is a non-trivial proper G-invariant partition then ρ is section-regular with respect to G. Thus we have the very simple observation that

> *if G is a synchronizing group then G is primitive on X,*

and Araújo's question is whether the converse is true.

This paper describes a preliminary study of the situation. It contains some general analysis, descriptions of a number of examples, a proof that, in quite a strong sense, for most n all primitive groups of degree n are synchronizing, and some