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Arithmetic of a Singular K3 Surface

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1. Introduction

This paper investigates the arithmetic of a particular singular K3 surface X over $\mathbb{Q}$, the extremal elliptic fibration with configuration $[1,1,1,12,3^*]$. First of all, we determine the corresponding weight-3 form (cf. [Li, Ex. 1.6]) explicitly. For this, we calculate the action of Frobenius on the transcendental lattice by counting points and applying the Lefschetz fixed point formula. The proof is based on our previous classification of complex multiplication (CM) forms with rational coefficients in [S1]. In fact, we only have to compute one trace.

Then we compute the zeta-function of the surface. This is used to study the reductions of X modulo some primes p. We emphasize that we are able to find a model with good reduction at 2. We subsequently verify conjectures of Tate and Shioda. The conjectures will be recalled in Section 4 and verified in Sections 5–7.

The final section is devoted to the twists of X. We show that these produce all newforms of weight 3 that have rational coefficients and CM by $\mathbb{Q}(\sqrt{-3})$.

2. The Extremal Elliptic K3 Fibration

There is a unique elliptic K3 surface X with a section and singular fibres I_1, I_1, I_1, I_{12}, and I_3^*. The configuration is listed as $[1,1,1,12,3^*]$ under No. 166 in [ShiZ] and [S3, Tab. 2]. Since the fibration arises as cubic base change of the extremal rational elliptic surface Y with singular fibres I_1^*, I_4, and I_1, we shall start by studying this surface.

In [MP], an affine Weierstrass equation of this fibration was given as

$$Y'\colon\ y^2 = x^3 - 3(s-2)^2(s^2-3)x + s(s-2)^3(2s^2-9). \tag{1}$$

It has discriminant

$$\Delta = 16 \cdot 27(s-2)^7(s+2),$$

so the singular fibres are I_1^* above 2, I_1 above -2, and I_4 above ∞.

We shall look for a model of Y over $\mathbb{Q}$ that has everywhere good reduction. The fibre of Y' at ∞ has nonsplit multiplicative reduction, so $\mathrm{H}^2_{\text{ét}}(Y', \mathbb{Q}_\ell)$ is ramified. Therefore we twist equation (1) over the splitting field $\mathbb{Q}(\sqrt{-3})$. Performing some elementary transformations (cf. [S2, IV.1]), we obtain the equation

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