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Almost-Minimal Nonuniform Lattices of Higher Rank

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1. Introduction

We find the minimal elements in three different (but essentially equivalent) partially ordered categories of mathematical objects:

(A) finite-volume, noncompact, complete, locally symmetric spaces of higher rank;

(B) nonuniform, irreducible lattices in semisimple Lie groups of higher real rank; and

(C) isotropic, simple algebraic $\mathbb{Q}$-groups of higher real rank.

The main interest is in categories (A) and (B), but the proof is carried out using the machinery of (C). (For completeness, we also provide a generalization that applies to algebraic groups over any number field, not only $\mathbb{Q}$.) Justification of the examples and facts stated in the Introduction can be found in Section 2.

1A. Locally Symmetric Spaces

It is well known that if G is a connected, semisimple Lie group and $\mathbb{R}$-rank $G \geq 2$, then G contains a closed subgroup that is locally isomorphic to either $\mathrm{SL}_3(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. Passing from semisimple Lie groups to the corresponding symmetric spaces yields the following geometric translation of this observation.

1.1. Fact. Let $\tilde{X}$ be a symmetric space of noncompact type, with no Euclidean factors, such that rank $\tilde{X} \geq 2$. Then $\tilde{X}$ contains a totally geodesic submanifold $\tilde{X}'$ such that $\tilde{X}'$ is the symmetric space associated to either $\mathrm{SL}_3(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. In other words, $\tilde{X}'$ is isometric to either

(1) $\mathrm{SL}_3(\mathbb{R})/\mathrm{SO}(3) \cong \{3 \times 3$ positive-definite symmetric real matrices of determinant $1\}$ or

(2) the product $\mathbb{H}^2 \times \mathbb{H}^2$ of two hyperbolic planes.

In short, among all the symmetric spaces of noncompact type with rank ≥ 2, there are only two manifolds that are minimal with respect to the partial order defined by totally geodesic embeddings. Our main theorem provides an analogue of

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