Michigan Math. J. 57 (2008)

Sally Modules of Rank One

Shiro Goto, Koji Nishida, & Kazuho Ozeki

1. Introduction

Let A be a Cohen–Macaulay local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. We assume the residue class field $k = A/\mathfrak{m}$ of A is infinite. Let I be an $\mathfrak{m}$-primary ideal in A and choose a minimal reduction $Q = (a_1, a_2, \dots, a_d)$ of I. Let

$$R = \mathrm{R}(I) := A[It] \quad \text{and} \quad T = \mathrm{R}(Q) := A[Qt] \subseteq A[t],$$

respectively, denote the Rees algebras of I and Q, where t stands for an indeterminate over A. We put

$$R' = \mathrm{R}'(I) := A[It, t^{-1}], \qquad T' = \mathrm{R}'(Q) := A[Qt, t^{-1}],$$

and

$$G = \mathrm{G}(I) := R'/t^{-1}R' \cong \bigoplus_{n \ge 0} I^n/I^{n+1}.$$

Let $B = T/\mathfrak{m}T$, which is the polynomial ring with d indeterminates over the field k. Following Vasconcelos [13], we then define

$$\mathrm{S}_Q(I) = IR/IT$$

and call it the *Sally module* of I with respect to Q. We observe that the Sally module $S = \mathrm{S}_Q(I)$ is a finitely generated graded T-module, since R is a module-finite extension of the graded ring T.

Let $\ell_A(\cdot)$ stand for the length and consider the Hilbert function

$$H_I(n) = \ell_A(A/I^{n+1})$$

$(n \ge 0)$ of I. Then we have the integers $\{\mathrm{e}_i = \mathrm{e}_i(I)\}_{0 \le i \le d}$ such that the equality

$$H_I(n) = \mathrm{e}_0\binom{n+d}{d} - \mathrm{e}_1\binom{n+d-1}{d-1} + \dots + (-1)^d \mathrm{e}_d$$

holds for all $n \gg 0$.

The Sally module S was introduced by Vasconcelos [13], where he gave an elegant review (in terms of his Sally module) of Sally's works [10; 11; 12] about the structure of $\mathfrak{m}$-primary ideals I with interaction to the structure of G and Hilbert coefficients e_i. Sally first investigated those ideals I satisfying the equality $\mathrm{e}_1 = \mathrm{e}_0 - \ell_A(A/I) + 1$ and gave several important results, among which one

Received July 25, 2007. Revision received October 4, 2007.