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Duality and Tameness

Marc Chardin, Steven Dale Cutkosky,
Jürgen Herzog, & Hema Srinivasan

Introduction

The purpose of this paper is to construct examples of strange behavior of local cohomology. In these constructions we follow a strategy that was already used in [CH] and that relates, via a spectral sequence introduced in [HRa], the local cohomology for the two distinguished bigraded prime ideals in a standard bigraded algebra.

In the first part we consider algebras with rather general gradings and deduce a similar spectral sequence in this more general situation. A typical example of such an algebra is the Rees algebra of a graded ideal. The proof for the spectral sequence given here is simpler than that of the corresponding spectral sequence in [HRa].

In the second part of this paper we construct examples of standard graded rings A, which are algebras over a field K, such that the function

$$j \mapsto \dim_K(H^i_{A_+}(A)_{-j}) \tag{1}$$

is an interesting function for $j \gg 0$. In our examples, this dimension will be finite for all j.

Suppose that A_0 is a Noetherian local ring and that $A = \bigoplus_{j\ge 0} A_j$ is a standard graded ring, and set $A_+ := \bigoplus_{j>0} A_j$. Let M be a finitely generated graded A-module and let $\mathcal{F} := \tilde{M}$ be the sheafification of M on $Y = \mathrm{Proj}(A)$. We then have graded A-module isomorphisms

$$H^{i+1}_{A_+}(M) \cong \bigoplus_{n\in\mathbf{Z}} H^i(Y, \mathcal{F}(n))$$

for $i \ge 1$ as well as a similar expression for $i = 0$ and 1.

By Serre vanishing, $H^i_{A_+}(M)_j = 0$ for all i and $j \gg 0$. However, the asymptotic behavior of $H^i_{A_+}(M)_{-j}$ for $j \gg 0$ is much more mysterious.

In the case when $A_0 = K$ is a field, the function (1) is in fact a polynomial for large enough j. The proof is a consequence of graded local duality ([BrS, 13.4.6] or [BH, 3.6.19]) and follows also from Serre duality on a projective variety.

For more general A_0, the $H^i_{A_+}(M)_{-j}$ are finitely generated A_0 modules but need not have finite length.

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