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Discreteness and Rationality of F-Thresholds

Manuel Blickle, Mircea Mustaţă, & Karen E. Smith

Dedicated to Mel Hochster on the occasion of his sixty-fifth birthday

1. Introduction

In recent years, multiplier ideals have played an increasingly important role in higher-dimensional birational geometry. For a given ideal $\mathfrak{a}$ on a smooth variety X and a real parameter $c > 0$, the multiplier ideal $\mathcal{J}(\mathfrak{a}^c)$ is defined via a log resolution of the pair $(X, \mathfrak{a})$. Recall that a log resolution is a proper birational map $\pi : X' \to X$, with X' smooth such that $\mathfrak{a}\mathcal{O}_{X'}$ defines a simple normal crossing divisor $A = \sum_{i=1}^{r} a_i E_i$. Then, by definition,

$$\mathcal{J}(\mathfrak{a}^c) := \pi_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor cA \rfloor), \tag{1}$$

and this is an ideal of $\mathcal{O}_X$ that does not depend on the chosen log resolution. A *jumping coefficient* (also called a jumping *number* or a jumping *exponent*) of $\mathfrak{a}$ is a positive real number c such that $\mathcal{J}(\mathfrak{a}^c) \neq \mathcal{J}(\mathfrak{a}^{c-\varepsilon})$ for every $\varepsilon > 0$. These invariants were introduced and studied in [ELSV]. It follows from formula (1) that, if c is a jumping coefficient, then $c \cdot a_i$ is an integer for some i. In particular, every jumping coefficient is a rational number, and the set of jumping coefficients of a given ideal is discrete.

Hara and Yoshida [HaY] introduced a positive characteristic analogue of multiplier ideals, denoted by $\tau(\mathfrak{a}^c)$. This is a generalized test ideal for a tight closure theory with respect to the pair $(X, \mathfrak{a}^c)$. Similarly, one can define jumping numbers for such test ideals. These invariants were studied under the name of *F-thresholds* in [MTW], where it was shown that they satisfy many of the formal properties of the jumping coefficients in characteristic 0.

We emphasize that the test ideals are not determined by a log resolution of singularities—even in cases where such a resolution is known to exist. Instead, the definition uses the Frobenius morphism and requires a priori infinitely many conditions to be checked. This lack of built-in finiteness makes the question of rationality and discreteness of the F-thresholds nontrivial, and in fact these properties were left open in [MTW].

In this paper we settle these questions in the case of a regular ring R that is essentially of finite type over an F-finite field. More precisely, we show that, for

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