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On Sections of Elliptic Fibrations

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1. Introduction

It is well known that two generic cubics P and Q in $\mathbb{C}P^2$ intersect each other in nine points $z_1, \dots, z_9$. By constructing the corresponding *pencil* of curves

$$\{sP + tQ \mid [s:t] \in \mathbb{C}P^1\}$$

one can define a map $f\colon \mathbb{C}P^2 - \{z_1, \dots, z_9\} \to \mathbb{C}P^1$. After blowing up $\mathbb{C}P^2$ at $\{z_1, \dots, z_9\}$ one can extend f to a Lefschetz fibration $\pi\colon E(1) = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2} \to \mathbb{C}P^1$ with nine distinguished sections and whose generic fiber is an elliptic curve. Our aim in this paper is to describe an analogous construction in the smooth category, but unfortunately we do not know whether our construction arises from an *algebraic* pencil of curves. Nevertheless, many 4-manifold topologists were curious about such a differential topological construction (e.g., this was posed explicitly as a question in [4]).

Let $\Gamma^s_{g,k}$ denote the mapping class group of a compact connected orientable genus-g surface with k boundary components and s marked points, so that diffeomorphisms and isotopies of the surface are assumed to fix the marked points and the points on the boundary. (We will drop k if the surface is closed and drop s if there are no fixed points.) A product $\prod_{i=1}^m t_i$ of right-handed Dehn twists in Γ_g provides a genus-g Lefschetz fibration $X \to D^2$ over the disk with closed fibers. If $\prod_{i=1}^m t_i = 1$ in Γ_g then the fibration closes up to a fibration over the sphere S^2. A lift of the relation $\prod_{i=1}^m t_i = 1$ to Γ^k_g shows the existence of k disjoint sections of the induced Lefschetz fibration. The self-intersection of the jth section is $-n_j$ if $\prod_{i=1}^m t_i = t_{\delta_1}^{n_1} \cdots t_{\delta_k}^{n_k}$ in $\Gamma_{g,k}$ for some positive integers $n_1, \dots, n_k$, where the t_{δ_i} are right-handed Dehn twists along circles parallel to the boundary components of the surface at hand (cf. [3]).

On the other hand, an expression of the form $\prod_{i=1}^m t_i = t_{\delta_1} \cdots t_{\delta_k}$ in $\Gamma_{g,k}$ naturally describes a Lefschetz pencil: the relation determines a Lefschetz fibration with k disjoint sections, where each section has self-intersection -1, and after blowing these sections down we get a Lefschetz pencil (cf. [4]). Conversely, blowing up the base locus of a Lefschetz pencil yields a Lefschetz fibration that can be captured

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