Michigan Math. J. 56 (2008)

Singularities Appearing on Generic Fibers of Morphisms between Smooth Schemes

Stefan Schröer

Introduction

The goal of this paper is to explore the structure of singularities that occur on generic fibers in positive characteristics. As an application of our general results, we shall determine which rational double points do and which do not occur on generic fibers.

In some sense, our starting point is Sard's lemma from differential topology. It states that the critical values of a differential map between differential manifolds form a set of measure 0. As a consequence, any general fiber of a differential map is itself a differential manifold. The analogy in algebraic geometry is as follows. Let k be an algebraically closed ground field of characteristic $p \geq 0$, and suppose $f: S \to B$ is a morphism between smooth integral schemes. Then the generic fiber S_η is a regular scheme of finite type over the function field $E = \kappa(\eta)$.

In characteristic 0, this implies that S_η is smooth over E. Moreover, the absolute Galois group $G = \mathrm{Gal}(\bar{E}/E)$ acts on the geometric generic fiber $S_{\bar{\eta}}$ with quotient isomorphic to S_η. In other words, to understand the generic fiber it suffices to understand the geometric generic fiber, which is again smooth over an algebraically closed field, together with its Galois action.

The situation is more complicated in characteristic $p > 0$. The reason is that over nonperfect fields the notion of regularity, which depends only on the scheme and not on the structure morphism, is weaker than the notion of geometric regularity, which coincides with formal smoothness. Here it easily happens that the geometric generic fiber $S_{\bar{\eta}}$ acquires singularities. This special effect already plays a crucial role in the extension of Enriques' classification of surface to positive characteristics: as Bombieri and Mumford [3] showed, there are *quasi-elliptic fibrations* for $p = 2$ and $p = 3$, which are analogous to elliptic fibrations but have a cusp on the geometric generic fiber.

We call a proper morphism $f: S \to B$ of smooth algebraic schemes a *quasi-fibration* if $\mathcal{O}_B = f_*(\mathcal{O}_S)$ and if the generic fiber S_η is not smooth. The existence of quasi-fibrations should by no means be viewed as pathological. Rather, they involve some fascinating geometry and apparently offer new freedom to achieve geometrical constructions that are impossible in characteristic 0. The theory of quasi-fibrations, however, is still in its infancy. In [15, Rem. 1.2], Kollár asks

Received October 2, 2006. Revision received June 29, 2007.