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On the Equation $\tau(\lambda(n)) = \omega(n) + k$

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1. Introduction

For every positive integer n, the function $\tau(n)$ counts the number of divisors of n, the function $\omega(n)$ counts the number of distinct prime divisors of n, and the Carmichael function $\lambda(n)$ is the exponent of the multiplicative group of the invertible congruence classes modulo n. The value of the function $\lambda(n)$ can be computed as follows:

$$\lambda(n) = \begin{cases} 1 & \text{if } n = 1; \\ 2^{\alpha-2} & \text{if } n = 2^\alpha,\ \alpha > 2; \\ p^{\alpha-1}(p-1) & \text{if } n = p^\alpha \text{ and } \begin{array}{l} p \geq 3 \text{ or} \\ p = 2,\ \alpha \leq 2; \end{array} \\ [\lambda(p_1^{\alpha_1}), \dots, \lambda(p_s^{\alpha_s})] & \text{if } n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}. \end{cases}$$

In [6], Erdős, Pomerance, and Schmutz proved a number of fundamental properties of λ. In the process of proving the lower bound $\lambda(n) > (\log n)^{c_0 \log\log\log n}$ for all large n (provided $c_0 < 1/\log 2$), they proved the inequality

$$n \leq (4\lambda(n))^{3\tau(\lambda(n))}.$$

Numerical calculations suggest that the stronger inequality

$$n \leq \lambda(n)^{\tau(\lambda(n))} \tag{1}$$

holds except for $n = 2, 6, 8, 12, 24, 80, 120, 240$. This will be proved in Corollary 1. One of the tools for proving (1) is the inequality $\tau(\lambda(n)) > \omega(n)$, which holds except for $n = 2, 6, 12, 24, 30, 60, 120, 240$; we will prove this in Proposition 1 and Proposition 2.

This motivates us to compare $\tau(\lambda(n))$ with $\omega(n)$. Since $\tau(\lambda(n)) \geq \omega(n)$ holds for all positive integers n (see Proposition 1), we can write $\tau(\lambda(n)) = \omega(n) + k$, where k is some nonnegative integer depending on n. We then fix $k \geq 0$ and investigate the positive integers n such that $\tau(\lambda(n)) = \omega(n) + k$.

Throughout this paper, we use x to denote a positive real number. We also use the Landau symbols O and o and the Vinogradov symbols $\gg$ and $\ll$ with their

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