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On Abel Maps of Stable Curves

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1. Introduction

We construct Abel maps for a stable curve X. Namely, for each one-parameter deformation of X to a smooth curve having regular total space and for each $d \geq 1$, we construct by specialization a map $\alpha_X^d : \dot{X}^d \to \overline{P_X^d}$, where $\dot{X} \subseteq X$ is the smooth locus and where $\overline{P_X^d}$ is the coarse moduli scheme for equivalence classes of degree-d "semibalanced" line bundles on semistable curves having X as a stable model. For $d = 1$, we show that α_X^1 extends to a map $\overline{\alpha_X^1} : X \to \overline{P_X^1}$ and does not depend on the choice of the deformation. Finally, we give a precise description of when $\overline{\alpha_X^1}$ is injective.

The theory of Abel maps for smooth curves goes back to the nineteenth century. In the modern language, let C be a smooth projective curve and let $\operatorname{Pic}^d C$ be its degree-d Picard variety parameterizing line bundles of degree d on C. For each $d > 0$ there exists a remarkable morphism, often called the dth *Abel map*:

$$\begin{array}{ccc} C^d & \longrightarrow & \operatorname{Pic}^d C, \\ (p_1, \dots, p_d) & \longmapsto & \mathcal{O}_C\left(\sum p_i\right). \end{array}$$

This map has been extensively studied and used in the literature. For $d = 1$ it gives, after the choice of a "base" point on C, the Abel–Jacobi embedding $C \hookrightarrow \operatorname{Pic}^1 C \cong \operatorname{Pic}^0 C$ (unless $C \cong \mathbb{P}^1$). For an interesting historic survey see [K1] or [K2].

What about Abel maps for singular curves? Abel maps were constructed for all integral curves in [AK] and were further studied in [EGK1; EGK2; EK]. In [AK] it is shown that the first Abel map of an integral singular curve is an embedding into its compactified Picard scheme. However, almost nothing is known for reducible curves—not even when they are stable. This lack of knowledge is all the more regrettable given the importance of stable curves in moduli theory.

In this paper we construct Abel maps for stable curves. As we see it, Abel maps should satisfy the following natural properties. First, they should have a geometric meaning. More explicitly, recall that for a smooth curve C the dth Abel map is the "moduli map" defined by a natural line bundle on $C^d \times C$; see Section 2.5. We want a similar property to hold for singular curves as well.

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