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On Normal K3 Surfaces

Ichiro Shimada

1. Introduction

In this paper, by a $K3$ surface we mean, unless otherwise stated, an *algebraic* $K3$ surface defined over an algebraically closed field.

A $K3$ surface X is said to be *supersingular* (in the sense of Shioda [23]) if the rank of the Picard lattice S_X of X is 22. Supersingular $K3$ surfaces exist only when the characteristic of the base field is positive. Artin [3] showed that, if X is a supersingular $K3$ surface in characteristic $p > 0$, then the discriminant of S_X can be written as $-p^{2\sigma_X}$, where σ_X is an integer with $0 < \sigma_X \leq 10$. This integer σ_X is called the *Artin invariant* of X.

Let Λ_0 be an even unimodular $\mathbb{Z}$-lattice of rank 22 with signature $(3, 19)$. By the structure theorem for unimodular $\mathbb{Z}$-lattices (see e.g. [16, Chap. V]), the $\mathbb{Z}$-lattice Λ_0 is unique up to isomorphisms. If X is a complex $K3$ surface, then $H^2(X, \mathbb{Z})$ regarded as a $\mathbb{Z}$-lattice by the cup product is isomorphic to Λ_0. For an *odd* prime integer p and an integer σ with $0 < \sigma \leq 10$, we denote by $\Lambda_{p,\sigma}$ an even $\mathbb{Z}$-lattice of rank 22 with signature $(1, 21)$ such that the discriminant group $\mathrm{Hom}(\Lambda_{p,\sigma}, \mathbb{Z})/\Lambda_{p,\sigma}$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{\oplus 2\sigma}$. Rudakov and Shafarevich [14, Sec. 1, Thm.] showed that the $\mathbb{Z}$-lattice $\Lambda_{p,\sigma}$ is unique up to isomorphisms. If X is a supersingular $K3$ surface in characteristic p with Artin invariant σ, then S_X is p-elementary by [14, Sec. 8, Thm.] and of signature $(1, 21)$ by the Hodge index theorem; hence S_X is isomorphic to $\Lambda_{p,\sigma}$.

The *primitive closure* of a sublattice M of a $\mathbb{Z}$-lattice L is $(M \otimes_{\mathbb{Z}} \mathbb{Q}) \cap L$, where the intersection is taken in $L \otimes_{\mathbb{Z}} \mathbb{Q}$. A sublattice $M \subset L$ is said to be *primitive* if $(M \otimes_{\mathbb{Z}} \mathbb{Q}) \cap L = M$ holds. For $\mathbb{Z}$-lattices L and L', we consider the following condition.

$$\mathrm{Emb}(L, L')\text{: There exists a primitive embedding of } L \text{ into } L'.$$

We denote by $\mathcal{P}$ the set of prime integers. For a nonzero integer m, we denote by $\mathcal{D}(m) \subset \mathcal{P}$ the set of prime divisors of m. We consider the following arithmetic condition on a nonzero integer d, a prime integer $p \in \mathcal{P} \setminus \mathcal{D}(2d)$, and a positive integer $\sigma \leq 10$.

$$\mathrm{Arth}(p, \sigma, d)\text{: } \left(\frac{(-1)^{\sigma+1} d}{p}\right) = -1,$$

where $\left(\frac{x}{p}\right)$ is the Legendre symbol.

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