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Rational Curves on Blowing-ups of Projective Spaces

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1. Introduction

Let $\mathrm{Mor}_\beta(\mathbb{P}^1, Y)$ denote the moduli space of morphisms f from a complex projective line $\mathbb{P}^1$ to a smooth complex projective variety Y such that $f_*[\mathbb{P}^1] = \beta$, where β is a given second homology class of Y. We study the irreducibility and the rational connectedness of the moduli space when Y is a successive blowing-up of a product of projective spaces with a suitable condition on β.

Before stating the Main Theorem proven in this paper, we introduce some notation. Let $X = \prod_{k=1}^m \mathbb{P}^{n_k}$, $X_0 = X$, and let $\pi_i : X_i \to X_{i-1}$ $(i = 1, \dots, r)$ be a blowing-up of X_{i-1} along a smooth irreducible subvariety Z_i. Let $E_i^t \subset X_r$ be the total transform $(\pi_i \circ \cdots \circ \pi_r)^{-1} Z_i$ of the exceptional divisor associated to Z_i, and let H_k be the divisor class coming from the hyperplane class of the kth projective space $\mathbb{P}^{n_k}$. Let $m_i = \#\{Z_j \mid j < i, (\pi_j \circ \cdots \circ \pi_r)^{-1}(Z_j) \supset E_i^t\}$. So general points of Z_i are the (m_i)th infinitesimal points of X. Denote by $\mathrm{Mor}_\beta(\mathbb{P}^1, X_r)^\sharp$ the open sublocus of $\mathrm{Mor}_\beta(\mathbb{P}^1, X_r)$ consisting of those f whose images do not lie on exceptional divisors: $f(\mathbb{P}^1) \not\subseteq E_i^t$ for all i.

Main Theorem. *Assume that* $\beta \cdot \bigl(\pi^* H_k - \sum_{i=1}^r (m_i + 1) E_i^t\bigr) \ge 0$ *for all k and that* $\beta \cdot E_i^t \ge 0$ *for all i, where* $\pi = \pi_1 \circ \cdots \circ \pi_r$.

(1) *The moduli space* $\mathrm{Mor}_\beta(\mathbb{P}^1, X_r)^\sharp$ *consists of free morphisms and is an irreducible smooth variety of expected dimension.*
(2) *If Z_i are rationally connected for all i, then a projective and birational model of* $\mathrm{Mor}_\beta(\mathbb{P}^1, X_r)^\sharp$ *is also rationally connected.*
(3) *The moduli space* $\mathrm{Mor}_\beta(\mathbb{P}^1, X_r)$ *is smooth, and* $\mathrm{Mor}_\beta(\mathbb{P}^1, X_r)^\sharp$ *is dense in* $\mathrm{Mor}_\beta(\mathbb{P}^1, X_r)$, *if one of the following conditions hold*:
   (a) *all $\pi(E_i^t)$ are points in X;*
   (b) *all centers Z_i are convex (i.e., $H^1(\mathbb{P}^1, g^* T_{Z_i}) = 0$ for any morphism $g : \mathbb{P}^1 \to Z_i$), and $\pi(E_i^t)$ are disjoint to $\pi(E_j^t)$ for any $i \neq j$.*

Note that the irreducibility (respectively, the rational connectedness of a projective, birational model) of the morphism space $\mathrm{Mor}_\beta(\mathbb{P}^1, X_r)$ implies the irreducibility of the moduli space of rational curves C with numerical condition $[C] = \beta$.

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