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Invariant Metrics and Distances on Generalized Neil Parabolas

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1. Introduction and Results

In the survey paper [3], the authors asked for an effective formula for the Carathéodory distance $c_{A_{2,3}}$ on the Neil parabola $A_{2,3}$ (in the bidisc). Such a formula was presented in a more recent paper by Knese [4]. To repeat the main result of [4], we recall that the Neil parabola is given by $A_{2,3} := \{(z, w) \in \mathbb{D}^2 : z^2 = w^3\}$, where $\mathbb{D}$ denotes the open unit disc in the complex plane. Then there is the natural parameterization $p_{2,3} \colon \mathbb{D} \to A_{2,3}$, $p_{2,3}(\lambda) := (\lambda^3, \lambda^2)$. Moreover, let ρ denote the Poincaré distance of the unit disc. Recall that

$$\rho(\lambda, \mu) := \frac{1}{2} \log \frac{1 + m_{\mathbb{D}}(\lambda, \mu)}{1 - m_{\mathbb{D}}(\lambda, \mu)},$$

where

$$m_{\mathbb{D}}(\lambda, \mu) := \left| \frac{\lambda - \mu}{1 - \lambda\bar{\mu}} \right|, \quad \lambda, \mu \in \mathbb{D}.$$

Let $\lambda, \mu \in \mathbb{D}$. Then Knese's result is

$$c_{A_{2,3}}(p_{2,3}(\lambda), p_{2,3}(\mu)) = \begin{cases} \rho(\lambda^2, \mu^2) & \text{if } |\alpha_0| \ge 1, \\ \rho\big(\lambda^2 \frac{\alpha_0 - \lambda}{1 - \bar{\alpha}_0 \lambda}, \mu^2 \frac{\alpha_0 - \mu}{1 - \bar{\alpha}_0 \mu}\big) & \text{if } |\alpha_0| < 1, \end{cases}$$

where $\alpha_0 := \alpha_0(\lambda, \mu) := \frac{1}{2}(\lambda + 1/\bar{\lambda} + \mu + 1/\bar{\mu})$. If $\lambda\mu = 0$ then the formula should be read as if $|\alpha_0| \ge 1$.

Observe that if λ and μ have a nonobtuse angle—that is, if $\operatorname{Re}(\lambda\bar{\mu}) \ge 0$—then $|\alpha_0(\lambda, \mu)| > 1$ (cf. Corollary 2).

Moreover, in [4] the formula for the Carathéodory–Reiffen pseudometric $\gamma_{A_{2,3}}$ is given as

$$\gamma_{A_{2,3}}((a, b); X) = \begin{cases} |X_2| & \text{if } a = b = 0 \text{ and } |X_2| \ge 2|X_1|, \\ \frac{4|X_1|^2 + |X_2|^2}{4|X_1|} & \text{if } a = b = 0 \text{ and } |X_2| < 2|X_1|, \\ \frac{2|\lambda b|}{1 - |b|^2} & \text{if } (a, b) \ne (0, 0) \text{ and } X = \lambda(3a, 2b),\ \lambda \in \mathbb{C}, \end{cases}$$

where $(a, b) \in A_{2,3}$ and $X \in T_{(a,b)} A_{2,3} :=$ the tangent space in (a, b) at $A_{2,3}$.

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