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Nonremovable Sets for Hölder Continuous Quasiregular Mappings in the Plane

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1. Introduction

Let $\alpha \in (0,1)$. A function $f: \mathbb{C} \to \mathbb{C}$ is said to be *locally α-Hölder continuous,* that is, $f \in \mathrm{Lip}_\alpha(\mathbb{C})$, if

$$|f(z) - f(w)| \le C|z - w|^\alpha \tag{1}$$

whenever $z, w \in \mathbb{C}$ and $|z - w| < 1$. A set $E \subset \mathbb{C}$ is said to be *removable* for α-Hölder continuous analytic functions if every function $f \in \mathrm{Lip}_\alpha(\mathbb{C})$, holomorphic on $\mathbb{C} \setminus E$, is actually an entire function. It turns out that there is a characterization of these sets E in terms of Hausdorff measures. For $\alpha \in (0,1)$, Dolženko [7] proved that a set E is removable for α-Hölder continuous analytic functions if and only if $\mathcal{H}^{1+\alpha}(E) = 0$. When $\alpha = 1$, we deal with the class of Lipschitz continuous analytic functions. Although the same characterization holds, a more involved argument, due to Uy [12], is needed to show that sets of positive area are not removable.

The same question may be asked in the more general setting of K-quasiregular mappings. Given a domain $\Omega \subset \mathbb{C}$ and $K \ge 1$, one says that a mapping $f: \Omega \to \mathbb{C}$ is *K-quasiregular in* Ω if f is a $W^{1,2}_{\mathrm{loc}}(\Omega)$ solution of the Beltrami equation

$$\bar\partial f(z) = \mu(z)\partial f(z)$$

for almost every $z \in \Omega$; here μ, the Beltrami coefficient, is a measurable function such that $|\mu(z)| \le \frac{K-1}{K+1}$ at almost every $z \in \Omega$. If f is a homeomorphism, then f is said to be *K-quasiconformal*. When $\mu = 0$, we recover the classes of analytic functions and conformal mappings on Ω, respectively.

It was shown in [6] that if E is a compact set satisfying $\mathcal{H}^d(E) = 0$ for $d = 2\frac{1+\alpha K}{1+K}$, then E is removable for α-Hölder continuous K-quasiregular mappings. This means that any function $f \in \mathrm{Lip}_\alpha(\mathbb{C})$, K-quasiregular in $\mathbb{C} \setminus E$, is actually K-quasiregular on the whole plane. To look for results in the converse direction, one observes that any compact set E with $\mathcal{H}^{1+\alpha}(E) > 0$ is nonremovable for holomorphic functions and hence also for K-quasiregular mappings in Lip_α. We are thus interested in dimensions between d and $1+\alpha$. In this paper we show that the index d is sharp in the following sense: Given $\alpha \in (0,1)$ and $K \ge 1$, for any $t > d$ there exist (i) a compact set E of dimension t and (ii) a function $f \in \mathrm{Lip}_\alpha(\mathbb{C})$

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