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Function Theory on the Neil Parabola

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1. Introduction

Distances on a complex space X that are invariant under biholomorphic maps have played an important role in the geometric approach to complex analysis. One of the oldest such distances is the the Carathéodory pseudodistance c_X ("pseudo" because the distance between two points can be zero). It was introduced by Carathéodory in 1926 and is extremely simple to define. The distance between two points x and y is defined to be the largest distance (using the Poincaré hyperbolic distance) that can occur between $f(x)$ and $f(y)$ under a holomorphic map f from X to the unit disk $\mathbb{D} \subset \mathbb{C}$. The Kobayashi pseudodistance k_X, introduced by Kobayashi in 1967, is defined in the opposite direction: the "distance" between two points x and y is now the infimum of the (hyperbolic) distance that can occur between two points $a, b \in \mathbb{D}$ for which there is a holomorphic map f from the disk to X mapping a to x and b to y. (Actually, there is a small technicality here; see Section 4 for the true definition.) A consequence of the Schwarz–Pick lemma on the disk (which says holomorphic self-maps of the disk are distance decreasing in the hyperbolic distance) is the fact that $c_X \leq k_X$.

For the purposes of motivating this paper, let us indulge in a short tangent. An interesting question—because of its geometric implications (including the existence of 1-dimensional analytic retracts)—is: For which complex spaces do we have $c_X = k_X$? The most important contribution to this question is by Lempert [11]. Lempert's theorem proves the Carathéodory and Kobayashi distances agree on a convex domain. This theorem came as a surprise for a couple of reasons: first, convexity is not a biholomorphic invariant; and second, which is our main point here, *there were not many explicit examples available at the time*. (The plot thickens on this problem: There is a domain—namely, the symmetrized bidisc—in $\mathbb{C}^2$ for which the two distances agree, yet this domain is not biholomorphically equivalent to a convex domain; see [9] for a summary of these results.) Although we cannot remedy the problem of a lack of examples in the past, we can attempt to add to the current selection of explicit examples. Many theorems about invariant metrics can be proved in the generality of complex spaces (see e.g. [10]) yet curiously there do not seem to be any nontrivial, explicit examples of the Carathéodory distance for a complex space *with a singularity*. Perhaps the

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