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Approximation of Plurisubharmonic Functions on Bounded Domains in $\mathbf{C}^n$

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1. Introduction

Let Ω be a domain in $\mathbf{C}^n$. An upper semicontinuous function $u : \Omega \to [-\infty, \infty)$ is said to be *plurisubharmonic* if the restriction of u to each complex line is subharmonic (we allow the function identically $-\infty$ to be plurisubharmonic). We write $\mathcal{PSH}(\Omega)$ for the set of plurisubharmonic functions on Ω, $\mathcal{PSH}^-(\Omega)$ for the set of plurisubharmonic functions bounded from above on Ω, and $\mathcal{PSH}(\bar\Omega)$ for the set of plurisubharmonic functions on neighborhoods of $\bar\Omega$. If u is a function bounded from above on Ω, then by u^* we mean the upper regularization of u; that is, if $z \in \bar\Omega$ then

$$u^*(z) = \limsup_{\xi \to z} u(\xi).$$

For a given point $z \in \bar\Omega$, we define the following class of Jensen measures:

$$J_z^1(\bar\Omega) = \left\{ \mu \in \mathcal{B}(\bar\Omega) : u^*(z) \le \int_{\bar\Omega} u^* \, d\mu \ \forall u \in \mathcal{PSH}^-(\Omega) \right\},$$

where $\mathcal{B}(\bar\Omega)$ is the set of positive regular Borel measures with mass 1 on $\bar\Omega$. We can define J_z^2 and J_z^3 analogously when $\mathcal{PSH}^-(\Omega)$ is replaced by $\mathcal{PSH}^c(\Omega)$ (the set of plurisubharmonic functions on Ω, continuous on $\bar\Omega$) and $\mathcal{PSH}^c(\bar\Omega)$ (the set of continuous functions on $\bar\Omega$ that are uniform limits of continuous functions in $\mathcal{PSH}(\bar\Omega)$), respectively. For simplicity of notation, we will write J_z^i instead of $J_z^i(\bar\Omega)$ if there is no risk of confusion. It is obvious that $\delta_z \in J_z^1 \subset J_z^2 \subset J_z^3$, where δ_z is the Dirac measure at z. With a little more effort, one can prove that each J_z^i is a closed convex subset of $\mathcal{B}(\bar\Omega)$. We say that Ω is *J-regular* if $J_z^1 = J_z^3$ for all $z \in \Omega$. The classes J_z^1, J_z^2, J_z^3 are introduced and studied extensively in [CCeW; DW; P; S2; W1; W2] and elsewhere. The main reason for introducing them is a duality theorem of Edwards that allows us to express upper envelopes of plurisubharmonic functions as lower envelopes of integrals with respect to Jensen measures. Since the traditional method of constructing plurisubharmonic functions has been to take envelopes over classes of plurisubharmonic functions, Edwards's duality theorem provides alternative ways of investigating these constructions. As an illustration of this idea, we prove in [DW] that, for every bounded domain Ω in $\mathbf{C}^n$: (i) if $J_z^1 = J_z^2$ for all $z \in \Omega$ then every $u \in \mathcal{PSH}^-(\Omega)$ is the pointwise limit

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