Michigan Math. J. 54 (2006)

Polynomial Convexity and Rossi's Local Maximum Principle

Jean-Pierre Rosay

Polynomial convexity is an old and fundamental topic in the theory of complex variables. If K is a compact set in $\mathbb{C}^n$ then its polynomial hull, denoted by $\hat{K}$, is the set of $z \in \mathbb{C}^n$ such that $|P(z)| \leq \mathrm{Sup}_K |P|$ for every (holomorphic) polynomial P. If $\hat{K} = K$ then K is said to be polynomially convex. It is my opinion that some very basic questions are still worth revisiting. I illustrate this with two examples.

In Part A, I discuss Rossi's local maximum principle. I noticed only recently that this principle becomes a totally trivial exercise if the hull is characterized in terms of plurisubharmonic functions. Rossi's principle then generalizes to almost complex manifolds.

In Part B, which is far less successful, I discuss the old result of polynomial convexity of (smooth enough) arcs. It is a deep result—still with no easy proof—and with an unsatisfactory conclusion, as will be explained later. I would like to see the polynomial convexity of arcs established by some kind of construction similar to the one in Part A (with the soft tool of plurisubharmonic functions). Part B is still far from that goal, but I hope the proof presented there is somewhat more pleasant than previous proofs. Some of its ingredients may be useful.

Part A

1. Introduction

Rossi's maximum principle [15] is an important result in complex analysis. One version reads as follows.

Theorem. *Let K be a compact set in $\mathbb{C}^n$, and let $\hat{K}$ be its polynomial hull. Let $z \in \hat{K} \setminus K$, and let V be a relatively compact neighborhood of z that does not intersect K. Then, for every polynomial P, $|P(z)| \leq \mathrm{Sup}_{\hat{K} \cap bV} |P|$ (where bV denotes the boundary of V).*

Thus, although the polynomial hull may not carry any analytic structure [16; 3, Chap. 24], still the maximum principle holds along $\hat{K}$. Following Rossi's original proof [15], proofs such as those for [3, Thm. 9.3] and [8, Thm. 3.2.11] rely on solving $\bar{\partial}$.

Received February 23, 2005. Revision received September 23, 2005.
Partly supported by an NSF grant.