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Power Structure over the Grothendieck Ring of Varieties and Generating Series of Hilbert Schemes of Points

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Introduction

The Grothendieck semiring $S_0(\mathcal{V}_{\mathbb{C}})$ of complex quasi-projective varieties is the semigroup generated by isomorphism classes $[X]$ of such varieties modulo the relation $[X] = [X - Y] + [Y]$ for a Zariski closed subvariety $Y \subset X$; the multiplication is defined by the Cartesian product: $[X_1] \cdot [X_2] = [X_1 \times X_2]$. The Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ is the group generated by these classes with the same relation and the same multiplication. Let $\mathbb{L} \in K_0(\mathcal{V}_{\mathbb{C}})$ be the class of the complex affine line, and let $K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]$ be the localization of Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ with respect to $\mathbb{L}$. A power structure over a (semi)ring R (as in [10]) is a map $(1 + T \cdot R[[T]]) \times R \to 1 + T \cdot R[[T]]$: $(A(T), m) \mapsto (A(T))^m$ $(A(T) = 1 + a_1T + a_2T^2 + \cdots, a_i \in R, m \in R)$ such that all usual properties of the exponential function hold. Over a ring R, a finitely determined (in a natural sense that we shall describe) power structure is defined by a pre-λ-ring structure on R (see [12]). Described in [10] is a power structure over each of the (semi)rings just defined. They are connected with the pre-λ-ring structure on the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ defined by the Kapranov zeta function [6; 11].

The main result of this paper is using the formalism of the power structure to express the generating series of classes (in the Grothendieck (semi)ring of varieties) of Hilbert schemes of zero-dimensional subschemes on a smooth quasi-projective variety of dimension d as an exponent of that for the complex affine space $\mathbb{A}^d$. The conjecture that the generating series of Hilbert schemes of points on a smooth surface can be considered as an exponent was communicated to the authors by D. van Straten. Specializations of this relation give formulas for generating series of certain invariants of the Hilbert schemes (Euler characteristic, Hodge–Deligne polynomial, ...).

We also describe a power structure over the ring $\mathbb{Z}[u_1, \dots, u_r]$ of polynomials in several variables with integer coefficients in such a way that, for $r = 2$, it is the specialization of the power structure over the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ under

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