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On the Homotopy Lie Algebra of an Arrangement

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1. Definitions and Statement of Results

1.1. Holonomy and Homotopy Lie Algebras

Fix a field $\Bbbk$ of characteristic 0. Let A be a graded, graded-commutative algebra over $\Bbbk$ with graded piece A_k, $k \geq 0$. We will assume throughout that A is locally finite, connected, and generated in degree 1. In other words, $A = T(V)/I$, where V is a finite-dimensional $\Bbbk$-vector space, $T(V) = \bigoplus_{k \geq 0} V^{\otimes k}$ is the tensor algebra on V, and I is a two-sided ideal generated in degrees 2 and higher. To such an algebra A, one naturally associates two graded Lie algebras over $\Bbbk$ (see e.g. [3; 14]).

Definition 1.1. The *holonomy Lie algebra* $\mathfrak{h}_A$ is the quotient of the free Lie algebra on the dual of A_1 modulo the ideal generated by the image of the transpose of the multiplication map $\mu\colon A_1 \wedge A_1 \to A_2$; thus,

$$\mathfrak{h}_A = \mathrm{Lie}(A_1^*)/\mathrm{ideal}(\mathrm{im}(\mu^*\colon A_2^* \to A_1^* \wedge A_1^*)). \tag{1}$$

Note that $\mathfrak{h}_A$ depends only on the quadratic closure of A: if we put $\bar{A} = T(V)/(I_2)$ then $\mathfrak{h}_A = \mathfrak{h}_{\bar{A}}$.

Definition 1.2. The *homotopy Lie algebra* $\mathfrak{g}_A$ is the graded Lie algebra of primitive elements in the Yoneda algebra of A:

$$\mathfrak{g}_A = \mathrm{Prim}(\mathrm{Ext}_A(\Bbbk, \Bbbk)). \tag{2}$$

In other words, the universal enveloping algebra of the homotopy Lie algebra is the Yoneda algebra:

$$U(\mathfrak{g}_A) = \mathrm{Ext}_A(\Bbbk, \Bbbk). \tag{3}$$

The algebra $U = \mathrm{Ext}_A(\Bbbk, \Bbbk)$ is a bigraded algebra; let us write U^{pq} to denote cohomological degree p and polynomial degree q. Then $U^{pq} = 0$ unless $-q \geq p$. The subalgebra $R = \bigoplus_{p \geq 0} U^{p,-p}$ is called the *linear strand* of U. For convenience we will let $U_q^p = U^{p,-p-q}$, where the lower index q is called the *internal*

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