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The Multipole Lempert Function Is Monotone under Inclusion of Pole Sets

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Let D be a domain in $\mathbb{C}^n$ and let $A = (a_j)_{j=1}^l$, $1 \le l \le \infty$, be a countable (i.e. $l = \infty$) or nonempty finite (i.e. $l \in \mathbb{N}$) subset of D. Moreover, fix a function $\boldsymbol{p}\colon D \to \mathbb{R}_+$ with

$$|\boldsymbol{p}| := \{a \in D : \boldsymbol{p}(a) > 0\} = A;$$

$\boldsymbol{p}$ is called a *pole function for* A on D and $|\boldsymbol{p}|$ its *pole set*. When $B \subset A$ is a nonempty subset we put $\boldsymbol{p}_B := \boldsymbol{p}$ on B and $\boldsymbol{p}_B := 0$ on $D \setminus B$. Then $\boldsymbol{p}_B$ is a pole function for B.

For $z \in D$ we set

$$l_D(\boldsymbol{p}, z) = \inf\left\{\prod_{j=1}^{l} |\lambda_j|^{\boldsymbol{p}(a_j)}\right\},$$

where the infimum is taken over all subsets $(\lambda_j)_{j=1}^l$ of $\mathbb{D}$ (in this paper, $\mathbb{D}$ is the open unit disc in $\mathbb{C}$) for which there is an analytic disc $\varphi \in \mathcal{O}(\mathbb{D}, D)$ with $\varphi(0) = z$ and $\varphi(\lambda_j) = a_j$ for all j. Here we call $l_D(\boldsymbol{p}, \cdot)$ the *Lempert function with* $\boldsymbol{p}$*-weighted poles at* A ([8; 9]; see also [5], where this function is called the *Coman function for* $\boldsymbol{p}$).

Wikström [8] has proved that, if A and B are finite subsets of a convex domain $D \subset \mathbb{C}^n$ with $\emptyset \ne B \subset A$ and if $\boldsymbol{p}$ is a pole function for A, then $l_D(\boldsymbol{p}, \cdot) \le l_D(\boldsymbol{p}_B, \cdot)$ on D.

On the other hand, in [9] Wikström gave an example of a complex space for which this inequality fails to hold, and he asked whether it remains true for an arbitrary domain in $\mathbb{C}^n$. The main purpose of this note is to present a positive answer to that question, even for countable pole sets. (In particular, it follows that the infimum in the definition of the Lempert function is always taken over a nonempty set.)

Theorem 1. *For any domain $D \subset \mathbb{C}^n$, any countable or nonempty finite subset A of D, and any pole function $\boldsymbol{p}$ for A, we have*

$$l_D(\boldsymbol{p}, \cdot) = \inf\{l_D(\boldsymbol{p}_B, \cdot) : \emptyset \ne B \text{ a finite subset of } A\}.$$

Therefore,

$$l_D(\boldsymbol{p}, \cdot) = \inf\{l_D(\boldsymbol{p}_B, \cdot) : \emptyset \ne B \subset A\}.$$

The proof of this result will be based on the following theorem.

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