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Residue Forms on Singular Hypersurfaces

Andrzej Weber

1. Introduction

The purpose of this paper is to point out a relation between the canonical sheaf and the intersection complex of a singular algebraic variety. We focus on the hypersurface case. Let M be a complex manifold and let $X \subset M$ be a singular hypersurface. We study residues of top-dimensional meromorphic forms with poles along X. Applying resolution of singularities, we are sometimes able to construct residue classes either in L^2-cohomology of X or in the intersection cohomology. The conditions that allow us to construct these classes coincide and can be formulated in terms of the weight filtration. Finally, provided that these conditions hold, we construct in a canonical way a lift of the residue class to the cohomology of X.

Let the manifold M be of dimension $n+1$. If the hypersurface X is smooth then we have an exact sequence of sheaves on M:

$$0 \hookrightarrow \Omega_M^{n+1} \hookrightarrow \Omega_M^{n+1}(X) \overset{\mathrm{Res}}{\twoheadrightarrow} i_*\Omega_X^n \longrightarrow 0.$$

Here Ω_M^{n+1} stands for the sheaf of holomorphic differential forms of the top degree on M and $\Omega_M^{n+1}(X)$ is the sheaf of meromorphic forms with logarithmic poles along X (i.e., with the poles of at most the first order). The map $i: X \hookrightarrow M$ is the inclusion. The morphism Res is the residue map sending $\omega = ds/s \wedge \eta$ to $\eta|_X$ if s is a local equation of X. The residues of forms with logarithmic poles along a smooth hypersurface were studied by Leray [Le] for forms of any degree. Later such forms and their residues were applied by Deligne ([D], see also [GS]) to construct the mixed Hodge structure for the cohomology of open smooth algebraic varieties.

We will allow X to have singularities. As in the smooth case, the residue form is a well-defined differential form on the nonsingular part of X. In general this form may be highly singular at the singular points of X. We will ask the following questions.

- Suppose M is equipped with a hermitian metric. Is the norm of $\mathrm{Res}(\omega)$ square integrable? We note that this condition does not depend on the metric.
- Does the residue form $\mathrm{Res}(\omega)$ define a class in the intersection cohomology $IH^n(X)$?

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