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On Fermat Curves and Maximal Nodal Curves

Mutsuo Oka

1. Introduction

Let $f(x) = a(x-\alpha_1)\cdots(x-\alpha_n)$, $a > 0$, be a real polynomial with n distinct real roots; it has $[(n-1)/2]$ maxima and $(n-1)-[(n-1)/2]$ minima. Thom has studied the space of real polynomials and showed, for example, that any given polynomial f can be deformed into a special polynomial that has the same maxima and minima [8]. A typical such polynomial is the Chebyshev polynomial.

A nodal curve C is an irreducible plane curve of degree n that contains only nodes ($= A_1$ singularities). A nodal curve is called a *maximal* nodal curve if it is rational and nodal; by Plücker's formula, it must contain $\frac{(n-1)(n-2)}{2}$ nodes to be maximal. In the space of polynomials of two variables, a maximal nodal curve can be understood as a generalization of a Chebyshev polynomial. In [6] the author constructed a maximal nodal curve of join type $f(x)+g(y)=0$ using a Chebyshev polynomial $f(x)$ and a similar polynomial $g(y)$ that has one maximal value and two minimal values.

In this paper we present another extremely simple way, for a given integer $n > 2$, to construct a beautifully symmetric and maximal nodal curve D_n as a by-product of the geometry of the Fermat curve $x^{n+1}+y^{n+1}+1=0$. A smooth point P of a plane curve C is called a *flex point of flex-order* $k \geq 3$ if the tangent line T_P at P and C intersect with intersection multiplicity k. The maximal nodal curve D_n, which we construct in this paper, contains three flexes of flex-order n, and it is symmetric with respect to the permutation of three variables U, V, W. By a special case of Zariski [10] and Fulton [2], $\pi_1(\mathbb{P}^2 - C) = \mathbb{Z}/n\mathbb{Z}$ if C is a maximal nodal curve of degree n. The examples D_n provide an alternate proof. Zariski [10] observed that the fundamental group of the complement of an irreducible curve C of degree n is abelian if C has a flex of flex-order either n or $n-1$. Since the moduli of maximal nodal curves of degree n is irreducible (by Harris [3]), the claim follows.

For the construction, we start from the Fermat curve $\mathcal{F}_n : x^n + y^n + 1 = 0$ and study singularities of the dual curve $\check{\mathcal{F}}_n$. The Fermat curve and the dual curve $\check{\mathcal{F}}_n$ have canonical $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ actions, so the defining polynomial of $\check{\mathcal{F}}_n$ is written as $h(u^n, v^n) = 0$ for a polynomial $h(u,v)$ of degree $n-1$. The curve $h(u,v)=0$ defines our maximal nodal curve D_{n-1}. Geometrically this is the quotient of the

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