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On Orientability and Degree of Fredholm Maps

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1. Introduction

The degree of a map between two manifolds has played important roles in various mathematical areas. Certain orientability is always required in order to make sense of the concept of degree. In the case of finite-dimensional nonorientable manifolds, this goes back to Hopf, Olum, and Steenrod, after Brouwer's pioneering work on orientable manifolds (cf. [9] and references therein). Elworthy and Tromba [4] took the first study in the case of infinite-dimensional Banach manifolds, where they introduced the degree on orientable Fredholm manifolds. This orientability restriction on manifolds is, however, often too severe and unnatural. It was Fitzpatrick, Pejsachowicz, and Rabier [6] who pointed out explicitly that the only requirement was the orientability of *maps* involved rather than that of *manifolds.* (The finite-dimensional version was in Olum's work.) Their approach is based on the concept of parity of paths, which makes it particularly useful in problems dealing with crossing singular strata. Indeed this is often the only practical way to check the orientability of a map. More recently, Benevieri and Furi [1] took another approach to orienting Fredholm maps that is conceptually more clear and seems more natural, since it comes directly from pointwise orientations of all Fredholm operators.

The approach taken in this paper has a more geometric flavor and also provides an instance where geometry and analysis interact nicely. The use of a determinant line bundle that arises from geometry links conveniently the notions of Benevieri–Furi and Fitzpatrick–Pejsachowicz–Rabier. In fact, many properties in [1], [2], and [6] become much easier to understand through our new approach. Conversely, the geometric approach allows us to apply functional analysis tools to some problems in gauge theory involving a real structure, where the relevant manifolds are often nonorientable or with no natural orientation, hence making it necessary to orient relevant maps instead. More details will appear in [10].

2. Fredholm Operator Families

We first consider orientability for families of Fredholm operators. To motivate the definition, we start with the case of a finite-dimensional manifold. Here orientability of the manifold can be characterized as the triviality of the orientation

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