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Solving the Schröder Equation at the Boundary in Several Variables

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0. Prologue

Let D be a domain of $\mathbb{C}$ containing the origin 0 and let $f \in \mathrm{Hol}(D, \mathbb{C})$ be a given holomorphic function defined on D with $f(0) = 0$ and $f'(0) \neq 0$. The classical Schröder equation [22] is the functional equation

$$\sigma \circ f = \lambda\sigma, \tag{0.1}$$

where σ is an unknown function, called a *Schröder map* for f, and λ is an unknown complex number.

This equation has a solution if, for instance, $|f'(0)| < 1$ (see Königs [16]). In particular, if $f \in \mathrm{Hol}(\Delta, \Delta)$ is a self-map (not an automorphism) of the unit disc $\Delta \subset \mathbb{C}$, then there exists a solution σ of (0.1) that is defined and holomorphic on Δ, and this solution is unique if the value of $\sigma'(0)$ is chosen. This last fact can be stated as the possibility of "representing" the given $f \in \mathrm{Hol}(\Delta, \Delta)$ by means of the linear part of its expansion at the fixed point 0.

If $f \in \mathrm{Hol}(\Delta, \Delta)$ has no fixed points in Δ, then the situation becomes more complicated and the existence of a solution σ of (0.1) depends on the "dynamics" of f. It is well known (see e.g. [1]) that, if $f \in \mathrm{Hol}(\Delta, \Delta)$ has no fixed points in Δ, then there exists a unique point $\tau \in \partial\Delta$ such that f has nontangential limit τ at τ—that is, τ is a boundary fixed point for f—and f' has nontangential limit $f'(\tau) = \alpha_\tau \in (0, 1]$ at τ. The real number α_τ is called the *boundary dilatation coefficient* of f at τ. If $\alpha_\tau < 1$ then the function f is called *hyperbolic,* otherwise it is called *parabolic.* If f is hyperbolic then (0.1) has a solution. The proof of this fact dates back to the times of Valiron [23]. If f is parabolic then there are no injective solutions of (0.1), and one is led to solve the so-called Abel's equation [3; 11; 19].

The Schröder (and Abel) equation can be written in a more general framework as follows. Let $f \in \mathrm{Hol}(\Delta, \Delta)$ and consider the equation

$$\sigma \circ f = \Phi_f \circ \sigma, \tag{0.2}$$

where Ω is a complex manifold, $\sigma\colon \Delta \to \Omega$ is an unknown holomorphic function (called an *intertwining map*), and Φ_f is a biholomorphism of Ω. A solution

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