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A Note on Mappings of Finite Distortion: The Sharp Modulus of Continuity

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1. Introduction

We consider Sobolev mappings $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, where Ω is a connected, open subset of $\mathbb{R}^n$ and $n \geq 2$. Thus, for almost every $x \in \Omega$, we can speak of the linear transform $Df(x) \colon \mathbb{R}^n \to \mathbb{R}^n$, called the differential of f at the point x. The Jacobian determinant $J(x, f)$ is the determinant of the matrix $Df(x)$: $J(x, f) = \det Df(x)$. We say that a mapping $f \colon \Omega \to \mathbb{R}^n$ has finite distortion if the following three conditions are satisfied:

(i) $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$;
(ii) the Jacobian determinant $J(x, f)$ of f is locally integrable; and
(iii) there is a measurable function $K_O = K_O(x) \geq 1$, finite almost everywhere, such that f satisfies the distortion inequality

$$|Df(x)|^n \leq K_O(x) J(x, f) \quad \text{a.e. } x \in \Omega. \tag{1}$$

Here we have used the operator norm of the differential matrix, defined by

$$|Df(x)| = \sup\{|Df(x)h| : |h| = 1\}.$$

We arrive at the usual definition of a mapping of bounded distortion, also called a quasiregular mapping, when we additionally require that $K_O \in L^\infty(\Omega)$. This class of mappings can be traced back to the work of Reshetnyak [12]. Mappings of bounded distortion are a natural generalization of analytic functions to higher dimensions. Undoubtedly, the theory of conformal mappings, or more generally of analytic functions, has also expanded in many other different directions.

In [12] Reshetnyak studied the continuity of mappings of bounded distortion. He proved that they are locally Hölder continuous with the exponent $1/K$, where K is the L^∞-norm of K_O. Here and in what follows, continuity for a Sobolev function f means that f can be modified in a set of Lebesgue measure zero to be continuous. For each constant $K \geq 1$, the radial stretching mapping

$$f(x) = x|x|^{1/K-1}$$

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