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Loop Structures on the Homotopy Type of S^3 Revisited

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1. Introduction

In an attempt to understand Lie groups from a homotopy theory point of view, Rector suggested studying Lie groups through their classifying spaces. Using S^3 as a test case, he proved in his pioneering paper [8] that there are uncountably many homotopically distinct deloopings of S^3. These deloopings form the so-called genus of the classifying space BS^3. To be more precise, for a nilpotent finite type space X, the *genus* of X is defined to be the set of homotopy types of nilpotent finite type spaces Y such that the p-completions of X and Y are homotopy equivalent for each prime p and also their rationalizations are homotopy equivalent. When considering genus, one often ignores the difference between a homotopy type and a space with that homotopy type.

Rector actually provided a complete list of classification invariants, defined by using integral and mod p cohomology, for the genus of BS^3, which we now call the Rector invariants. Briefly, the Rector invariants of a space X in the genus of BS^3 are signs, $(X/p) \in \{\pm 1\}$, one for each prime p. Two spaces in the genus of BS^3 are homotopy equivalent if and only if they have the same corresponding Rector invariants, and any such sequence does occur for some space. Also, BS^3 itself has 1 as its Rector invariant for each prime.

Generalizing the approach used by Rector, Møller [6] showed that this property of having a huge genus is not restricted to BS^3. In fact Møller proved that, whenever G is a compact connected non-abelian Lie group, the genus of its classifying space BG is uncountably large. There is, however, no explicit list of classification invariants (like the Rector invariants) in this general setting. A remarkable result of Notbohm [7] showed that K-theory ring together with the λ-operations classify the genus of BG, provided G is a simply connected compact Lie group. In other words, two spaces in the genus of BG are homotopy equivalent if and only if their K-theory are isomorphic as λ-rings. Notbohm's proof does not involve computing the K-theory of these spaces, and consequently we do not know how they are mutually nonisomorphic. This prompts the question of *how* these uncountably many λ-rings, the K-theory of the spaces in the genus of a given classifying space BG, are mutually nonisomorphic.

Ideally, one would like to compute explicitly these K-theory λ-rings, at least partially. Then one can use this knowledge to show that these λ-rings are mutually

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