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Families of Diffeomorphisms without Periodic Curves

Javier Ribón

1. Introduction

A formal curve is a reduced principal ideal $\hat{\gamma} = (\hat{f})$ of $\mathbb{C}[[x, y]]$. We say that $\hat{\gamma}$ is invariant by $\varphi \in \mathrm{Diff}(\mathbb{C}^2, 0)$ if

$$(\hat{f}) = (\hat{f}) \circ \varphi$$

and that $\hat{\gamma}$ is p-periodic if it is invariant by $\varphi^{(p)}$ and not by $\varphi^{(j)}$ for $0 < j < p$.

Theorem 1. *There exists a germ of a diffeomorphism $\varphi \in \mathrm{Diff}(\mathbb{C}^2, 0)$ that has no convergent periodic germs of curve.*

Moreover, we may choose φ inside each of the following classes.

- The formally linearizable germs of diffeomorphism.
- The germs of diffeomorphism whose linear part is the identity.

These germs of diffeomorphism have formal invariant curves. Although there are germs of diffeomorphism without formal invariant curves, we prove the following.

Theorem 2. *A formal diffeomorphism $\varphi \in \widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ has at least one irreducible formal periodic curve.*

In [6] Hakim exhibits germs of diffeomorphisms of the type

$$\rho_\alpha = \left(\frac{x}{1+x}, \; ye^{-\alpha x} + x^2 \right) \quad (\alpha \in \mathbb{C})$$

with divergent "strong" invariant curves, showing in this way that the argument in [2] does not work for germs of diffeomorphism. Nevertheless, the germs ρ_α preserve the foliation $dx = 0$, and this is essential in [6] for the proof of the divergence property. All the ρ_α have $x = 0$ as a periodic curve (in fact, a curve of fixed points). This situation is general, as our next result shows.

Theorem 3. *Let φ be an element of $\widehat{\mathrm{Diff}}(\mathbb{C}^2, 0)$ preserving a formal 1-dimensional foliation $\mathcal{F}$. Then there exists a formal curve γ that is periodic by φ and invariant by $\mathcal{F}$. If $\mathcal{F}$ is convergent then γ can be chosen to be convergent.*

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