Michigan Math. J. 53 (2005)

SAGBI Bases and Degeneration of Spherical Varieties to Toric Varieties

Kiumars Kaveh

Introduction

Let $X \subset \mathbb{P}(V)$ be a (normal) complex projective G-variety, where G is a (reductive) classical group and V is a complex finite-dimensional G-module. Suppose X is *spherical*—that is, a Borel subgroup has a dense orbit. Generalizing the case of toric varieties, one can associate an integral convex polytope $\Delta(X)$ to X such that the Hilbert polynomial $h(t)$ of X is the Ehrhardt polynomial of $\Delta(X)$, that is, $h(t) =$ number of integral points in $t\Delta(X)$. The polytope $\Delta(X)$ is the polytope fibred over the moment polytope of X with the Gelfand–Cetlin polytopes as fibres. This polytope was defined by Okounkov [O1] based on results of Brion. Following Okounkov, we call this polytope the *Newton polytope* of X.

In this paper, for $G = \mathrm{SP}(2n, \mathbb{C})$ we show that X can be deformed (degenerated), by a flat deformation, to the toric variety corresponding to the polytope $\Delta(X)$ (Corollary 5.5). This is a consequence of the main result of our paper: the homogeneous coordinate ring of a *horospherical* variety has a SAGBI basis (Theorem 5.1). A spherical variety is horospherical if the stabilizer of a point in the dense G-orbit contains a maximal unipotent subgroup. Flag varieties and Grassmanians are examples of horospherical varieties. It is known that any spherical variety can be deformed, by a flat deformation, to a horospherical variety such that the moment polytopes of the two varieties are the same (see [P; ABr, Sec. 2.2; Kn, Satz 2.3]).

More precisely, we prove that if $X \subset \mathbb{P}(V)$ is a projective horospherical G-variety ($G = \mathrm{SP}(2n, \mathbb{C})$), then the homogeneous coordinate ring R of X can be embedded in a Laurent polynomial algebra and has a SAGBI basis with respect to a natural term order. (SAGBI stands for subalgebra analogue of Gröbner basis for ideals.) Moreover, we show that the semi-group of initial monomials is the semi-group of integral points in the cone over the polytope $\Delta(X)$. A finite collection $f_1, \dots, f_r$ of elements of R is a SAGBI basis, with respect to a term order, if the semi-group of initial monomials is generated by the initial monomials of the f_i and if, moreover, every element of R can be represented as a polynomial in the f_i, in a finite number of steps, by means of a simple classical algorithm called the *subduction algorithm*.

Degenerations of flag and Schubert varieties to toric varieties have been studied by Gonciulea and Lakshmibai [GoL] and Caldero [Ca]. More recently, Kogan and

Received October 8, 2003. Revision received June 17, 2004.