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The Trace of an Automorphism on $H^0(J, \mathcal{O}(n\Theta))$

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1. Introduction

Let X be a projective smooth complex curve with group of automorphisms G. Let J be the Jacobian of X and let Θ be the theta divisor of J. Then G acts on J and Θ is invariant under the action of G. Given $h \in G$, our goal is to compute the trace of h on $H^0(J, \mathcal{O}(n\Theta))$ in order to decompose this space into a sum of irreducible representations of G. Dolgachev computed the decomposition of $H^0(J, \mathcal{O}(2\Theta))$ when X is the Klein quartic and used it to study some invariant vector bundles on this curve; see the proof of Corollary 6.3 in [4].

The strategy is as follows. Consider the exact sequence

$$0 \to \mathcal{O}(n\Theta) \to \mathcal{O}((n+1)\Theta) \to \mathcal{O}_\Theta((n+1)\Theta) \to 0. \tag{1}$$

By the Kodaira vanishing theorem we have

$$H^0(J, \mathcal{O}((n+1)\Theta)) = H^0(J, \mathcal{O}(n\Theta)) \oplus H^0(\Theta, \mathcal{O}((n+1)\Theta))$$

for $n \geq 1$. Then, all we need to do is to compute the decomposition for $H^0(\Theta, \mathcal{O}(n\Theta))$. The problem can be reduced to work with $H^0(S^{g-1}X, K_{S^{g-1}X}^{\otimes n})$, where $S^{g-1}X$ is the $g-1$ symmetric product of X, g is the genus of X, and $K_{S^{g-1}X}$ is the canonical line bundle of $S^{g-1}X$ (see Lemma 2.3). Now, to compute the trace of $h \in G$, we use the holomorphic Lefschetz theorem. There is no problem in applying this theorem if $\langle h \rangle \setminus \{1\}$ is contained in a conjugacy class of G (see Proposition 3.2), and the problem in general is how to compute the characteristic classes required in the theorem. If the fixed point set of h in $S^{g-1}X$ is finite, then it is still possible to compute the trace of h. If the components of the fixed point set of h in $S^{g-1}X$ have dimension at most 1, then by studying the function field of X one could proceed as in the example of [14] to compute the characteristic classes. We do not need to do the last in our examples; in fact, we have written a *Maple* program to compute the trace of h on $H^0(S^bX, K_{S^bX}^{\otimes n})$ when $\langle h \rangle \setminus \{1\}$ is contained in a conjugacy class of G. The program was used in our examples and can be obtained from me upon request. Our main results are Theorem 3.3 and the decomposition of $H^0(J, \mathcal{O}(n\Theta))$ for the Klein quartic, the Macbeath curve of genus 7, and the Bring curve of genus 4. This work is based on results from my thesis [13].

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