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The Minimal Marked Length Spectrum of Riemannian Two-Step Nilmanifolds

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Introduction

The purpose of this paper is to compare the minimal marked length spectrum and the Laplace spectrum on functions and on forms for Riemannian two-step nilmanifolds. A Riemannian nilmanifold is a closed manifold of the form $(\Gamma\backslash G, g)$, where G is a simply connected nilpotent Lie group, Γ is a cocompact (i.e., $\Gamma\backslash G$ compact) discrete subgroup of G, and g arises from a left invariant metric on G. Examples of nilmanifolds include flat tori and Heisenberg manifolds. The Laplace spectrum of a closed Riemannian manifold (M, g) is the set of eigenvalues of the Laplace–Beltrami operator Δ, counted with multiplicity. The Laplace–Beltrami operator may be extended to act on smooth p-forms by $\Delta = d\delta + \delta d$, where δ is the metric adjoint of d. Two manifolds have the same marked length spectrum if there exists an isomorphism between the fundamental groups such that corresponding free homotopy classes of loops can be represented by smoothly closed geodesics of the same length. Two manifolds have the same minimal marked length spectrum (resp., maximal marked length spectrum) if there exists an isomorphism between the fundamental groups such that the smallest (resp., longest) closed loops in corresponding free homotopy classes have the same length.

The main theorem of this paper is the following (see Theorem 2.5).

Theorem 1. *For a generic class of two-step nilmanifolds, if a pair of nilmanifolds in this class has the same minimal marked length spectrum, then the nilmanifolds necessarily share the same Laplace spectrum on functions and on forms and must also have the same marked length spectrum.*

We prove Theorem 2.5 by showing that the mapping that induces the marking between the fundamental groups must take the form of an almost inner automorphism composed with an isomorphism that is also an isometry. Work of Gordon and Wilson [GW1; G1] shows that almost inner automorphisms preserve the marked length spectrum and also preserve the Laplace spectrum on functions; DeTurck and Gordon [DG] showed that the Laplace spectrum on forms is preserved in this case.

We also prove the result without the generic hypothesis (see Theorem 4.1) in the class of nilmanifolds with a two-dimensional center. See Remark 3.4 for

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