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Intersective Sets and Diophantine Approximation

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1. Introduction

In 1939, Koksma [16] introduced a classification of the real transcendental numbers ξ in terms of the quality of their algebraic approximations. For any positive integer n, denote by $w_n^*(\xi)$ the supremum of the real numbers w for which there exist infinitely many real algebraic numbers α of degree at most n satisfying

$$0 < |\xi - \alpha| \leq \mathrm{H}(\alpha)^{-w-1},$$

where $\mathrm{H}(\alpha)$ is the naïve height of α, that is, the maximum of the absolute values of the coefficients of its minimal defining polynomial over the integers. Following Koksma, set

$$w^*(\xi) = \limsup_{n \to +\infty} \frac{w_n^*(\xi)}{n}$$

and call ξ an

S^*-number if $w^*(\xi) < +\infty$;

T^*-number if $w^*(\xi) = +\infty$ and $w_n^*(\xi) < +\infty$ for any $n \geq 1$;

U^*-number if $w^*(\xi) = +\infty$ and $w_n^*(\xi) = +\infty$ from some n onward.

It turns out (see e.g. Schneider [20]) that this classification coincides with that of Mahler introduced in 1932 [17], which depends on the accuracy with which nonzero integer polynomials evaluated at ξ approach zero. Sprindžuk [21] proved that almost all real numbers (in the sense of Lebesgue measure) are S^*-numbers and, moreover, satisfy $w_n^*(\xi) = n$ for any positive integer n. Using this result and the theory of Hausdorff dimension, Baker and Schmidt [1] established that, for any $n \geq 1$, the function w_n^* takes any value in the range $[n, +\infty[$ and even that, for any $\tau \geq 1$,

$$\dim\{\xi \in \mathbf{R} : w_n^*(\xi) \geq \tau(n+1) - 1\} = 1/\tau, \tag{1}$$

$$\dim\{\xi \in \mathbf{R} : w_n^*(\xi) = \tau(n+1) - 1\} = 1/\tau, \tag{2}$$

and

$$\dim \bigcap_{n \geq 1} \{\xi \in \mathbf{R} : w_n^*(\xi) \geq \tau(n+1) - 1\} = \frac{1}{\tau}, \tag{3}$$

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