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On the Gehring–Hayman Property, the Privalov–Riesz Theorems, and Doubling Measures

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1. Introduction and Main Results

The motivation for the results in this paper is threefold. First, in [BKR] the authors observed that a number of relevant results on conformal mapping rely on only two properties of the derivative $|f'|$ of a conformal map f of the unit disk in the complex plane: the *Harnack property* (H) and the so-called *volume growth property* (VG).

Let $\rho\colon \mathbb{R}^{N+1}_+ \to (0, +\infty)$ be a continuous function (a *metric density*) in the upper half-space. We say that ρ satisfies Harnack's property (H) with constant C if, for each $z = (a,t) \in \mathbb{R}^{N+1}_+$,

$$C^{-1} \le \frac{\rho(w)}{\rho(q)} \le C$$

whenever $w, q \in B\big(z, \frac{1}{2}t\big)$. (See the end of this section for notation.)

Associated to a metric density ρ in $\mathbb{R}^{N+1}_+$, we define the ρ-length of a curve Γ in $\mathbb{R}^{N+1}_+$ as

$$\text{length}_\rho(\Gamma) = \int_\Gamma \rho(z)\,|dz|$$

and the ρ-distance

$$d_\rho(w,q) = \inf_\Gamma\ \text{length}_\rho(\Gamma)$$

for $w, q \in \mathbb{R}^{N+1}_+$, where the infimum is taken over all curves in $\mathbb{R}^{N+1}_+$ joining w, q. Then d_ρ is a distance in $\mathbb{R}^{N+1}_+$. If $\rho(x,t) = 1/t$, then d_ρ is the *hyperbolic distance* in $\mathbb{R}^{N+1}_+$. We recall that the hyperbolic geodesics are exactly the vertical lines and the circles ending orthogonally at the boundary.

If $z \in \mathbb{R}^{N+1}_+$ and $r > 0$ then $B_\rho(z,r)$ denotes the open ball of center z and radius r in the distance d_ρ. We say that ρ satisfies the *volume growth* condition (VG) with constant C if

$$\mu_\rho(B_\rho(z,r)) \le C r^{N+1}$$

for all $z \in \mathbb{R}^{N+1}_+$ and $r > 0$, where μ_ρ is the volume measure associated to ρ.

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