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On the Problem of Kähler Convexity in the Bergman Metric

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1. Introduction

Let (M, ds^2) be a complete Kähler manifold of dimension n, and let $\mathcal{H}^{p,q}_{(2)}(M)$ be the space of square-integrable harmonic forms of bidegree (p,q). McNeal has studied the question: Under which reasonable conditions about the Kähler metric can one prove the vanishing of $\mathcal{H}^{p,q}_{(2)}(M)$ when $p+q \neq n$? As a sufficient condition he found that there should exist an exhausting function V for M that is at the same time a potential for ds^2 such that V dominates its gradient. We define this property as follows.

Definition. Assume that the Kähler metric ds^2 has a global potential $V \in C^2(M)$ on M. Then we say that V *dominates* its gradient if there exist constants $A, B \geq 0$ such that

$$|\partial V|^2_{ds^2} \leq A + BV \tag{1.1}$$

throughout M.

In [M2] such a Kähler manifold is called *Kähler convex*; if (1.1) holds with $B = 0$, it is called *Kähler hyperbolic.*

In complex analysis there is a case of special interest in which $M = D$ is a pseudoconvex bounded domain in $\mathbb{C}^n$ that is endowed with the Bergman metric. Let $K_D(z)$ denote the Bergman kernel function on the diagonal of $D \times D$. Then $V_D = \log K_D$ is a potential of the Bergman metric.

Donnelly and Fefferman [DoFe] proved the vanishing of $\mathcal{H}^{p,q}_{(2)}(D)$ when $p + q \neq n$ and D is strongly pseudoconvex. Later, Donnelly [Do1; Do2] gave a simpler proof of this by a method that applies also to the case of finite-type pseudoconvex domains in $\mathbb{C}^2$ and to certain classes of finite-type domains in $\mathbb{C}^n$ with $n \geq 3$ (see e.g. [M1]). In these cases he showed using results of [C; M1] that even Kähler hyperbolicity holds. Also in [Do2] it was shown that the domain $D = \{z \in \mathbb{C}^3 \mid |z_1|^2 + |z_2|^{10} + |z_3|^{10} + |z_2|^2|z_3|^2 < 1\}$ is not Kähler hyperbolic in the Bergman metric.

The purpose of this paper is to show (by means of an example) that, on a smooth bounded weakly pseudoconvex domain of finite type, the potential V_D in general will not dominate its gradient. We will do this using ideas from [Do2; M2]; the

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