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Lagrangian Surfaces with Circle Symmetry in the Complex Two-Space

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Introduction

The hyper-Kähler structure of the complex 2-space $\mathbb{C}^2$ produces an interesting correspondence between minimal Lagrangian surfaces and holomorphic curves. Indeed, the correspondence is given by exchanging the orthogonal complex structure J to another one on the Euclidean 4-space $\mathbb{R}^4$ [ChMo]. More generally, every Lagrangian conformal immersion of a Riemann surface M into $\mathbb{C}^2$ can be transformed to a map from M (possibly from a covering of M) satisfying a Dirac-type equation with a specific potential term. It is an extension of the Cauchy–Riemann equation. This result is based on the following obvious facts. First, every immersion of M into $\mathbb{C}^2$ is canonically identified with a section of the product vector bundle $M \times \mathbb{C}^2$ over M. Second, the "plus" part of the spin bundle $\mathbb{S}$ (associated to the canonical spin$^{\mathbb{C}}$-structure induced from the complex structure on M) is isomorphic to the product complex line bundle $M \times \mathbb{C}$. In fact, the operator associated with the Dirac-type equation is the bona fide Dirac operator on the direct sum of the spin bundle $\mathbb{S}$ and its conjugate spin bundle $\bar{\mathbb{S}}$ (see Section 1). Incidentally, it is now known that every conformal immersion of a Riemann surface into $\mathbb{R}^4$ or $\mathbb{R}^3$ can be expressed by a quaternionic analogue of the Cauchy–Riemann equation, and the quaternionic approach has advanced study for the surface theory (see [BFLPP]). However, we will study Lagrangian surfaces with prescribed Lagrangian angle in $\mathbb{C}^2$, and description in terms of the complex numbers seems to be suitable for this study.

In previous papers [A1; A2] we gave explicitly the transformation and Dirac-type equation just described in terms of the Lagrangian angle. For a given Lagrangian conformal immersion $f: M \to \mathbb{C}^2$, the Lagrangian angle β is defined as an $(\mathbb{R}/2\pi\mathbb{Z})$-valued function on M. It presents the self-dual part of the generalized Gauss map as for the surface in $\mathbb{R}^4$ ($\cong \mathbb{C}^2$). The mean curvature vector is given by $J\nabla\beta$. In particular, the angle β is constant if f is a minimal immersion. We proved that, if the half $\beta/2$ of Lagrangian angle is also well-defined globally on M, then f is represented by a solution of the Dirac-type equation on M with potential term of the complex derivative of β. (The corresponding results in [A1; A2] are not correct without the assumption on β.)

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