Michigan Math. J. 52 (2004)

Series of Lie Groups

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1. Introduction

One way to define a collection of Lie algebras $\mathfrak{g}(t)$, parameterized by t and each equipped with a representation $V(t)$, as forming a "series" is to require (following Deligne) that the tensor powers of $V(t)$ decompose into irreducible $\mathfrak{g}(t)$-modules in a manner independent of t, with formulas for the dimensions of the irreducible components of the form $P(t)/Q(t)$ where P, Q are polynomials that decompose into products of integral linear factors. We study such decomposition formulas in this paper, which provides a companion to [15], where we study the corresponding dimension formulas. We connect the formulas to the geometry of the closed orbits $X(t) \subset \mathbb{P}V(t)$ and their unirulings by homogeneous subvarieties. We relate the linear unirulings to work of Kostant [12]. By studying such series, we determine new modules that, appropriately viewed, are *exceptional* in the sense of Brion [2] (see, e.g., Theorem 6.2).

The starting point of this paper was the work of Deligne et al. [4; 6; 7] containing uniform decomposition and dimension formulas for the tensor powers of the adjoint representations of the exceptional simple Lie algebras up to $\mathfrak{g}^{\otimes 5}$. Deligne's method for the decomposition formulas was based on comparing Casimir eigenvalues, and he offered a conjectural explanation for the formulas via a categorical model based on bordisms between finite sets. Vogel [23] obtained similar formulas for all simple Lie superalgebras based on his *universal Lie algebra*. We show that all *primitive* factors in the decomposition formulas of Deligne and Vogel can be accounted for using a pictorial procedure with Dynkin diagrams. (The nonprimitive factors are those either inherited from lower degrees or arising from a bilinear form, so knowledge of the primitive factors gives the full decomposition.) We also derive new decomposition formulas for other series of Lie algebras.

In Section 2, we describe a pictorial procedure using Dynkin diagrams for determining the decomposition of $V^{\otimes k}$. In Sections 3 and 4 we distinguish and interpret the primitive components in the decomposition formulas of Deligne and Vogel.

The exceptional series of Lie algebras occurs as a line in Freudenthal's magic square (see e.g. [10] or the variant we use in [15]). The three other lines each come with preferred representations. Dimension formulas for all representations

Received May 13, 2003. Revision received March 4, 2004.