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Noncompact Codimension-1 Real Algebraic Manifolds

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1. Introduction and the Main Theorem

A classical theorem of Seifert asserts that every smooth, closed, codimension-1 submanifold of Euclidean n-space can be smoothly isotoped to a nonsingular real algebraic set [S, Satz 4]. We consider the noncompact analogue of Seifert's theorem.

The main result (Theorem 1) is a necessary and sufficient topological condition for X^n a smooth compact manifold with boundary to have a codimension-1, real algebraic interior. In particular, for such an X^n, there is a smooth proper embedding $X^n \hookrightarrow D^{n+1}$ if and only if the interior of X^n is diffeomorphic to a nonsingular real algebraic subset of $\mathbb{R}^{n+1}$. Moreover, if such an embedding exists, then $\operatorname{int}(X^n)$ is isotopic to a nonsingular real algebraic subset of $\operatorname{int}(D^{n+1}) \approx \mathbb{R}^{n+1}$.

Using Theorem 1 we show (Corollary 1) that the noncompact analogue of Seifert's theorem is intimately related to completions of a pair. This observation yields a complete answer to the noncompact Seifert problem in ambient dimension < 4 and also in the high-dimensional simply connected case (Corollary 2). As a final application of Theorem 1, we show that a real algebraic problem of V. I. Arnol̆d concerning exotic $\mathbb{R}^4$s being real algebraic in $\mathbb{R}^5$ is in fact equivalent to an open topological problem.

A guiding problem here is the noncompact analogue of Seifert's classical theorem [S, Satz 4].

Problem 1. Which smooth, proper, codimension-1 submanifolds M^n (not necessarily compact) of $\mathbb{R}^{n+1}$ are isotopic to nonsingular real algebraic sets?

The method of proof employed by Seifert in the compact case does not readily extend to the noncompact case. The main difference is the amount of control needed near infinity. In the compact case, one approximates a suitable smooth function by a polynomial over a large compact set and then, following Seifert, adds an algebraic correction term to keep the polynomial from picking up more zeros outside of the compact set. The only control needed near infinity is that the polynomial should always be greater than (or always less than) zero. Clearly much more control is needed in the proper noncompact case, since the set of zeros extends all the way to infinity. The noncompact case does eventually involve approximating

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