Michigan Math. J. 52 (2004)

Simplicial Intersections of a Convex Set and Moduli for Spherical Minimal Immersions

Gabor Toth

1. Introduction and Statement of Results

Let $\mathcal{H}$ be a Euclidean vector space. Let $S_0^2(\mathcal{H})$ denote the space of symmetric endomorphisms of $\mathcal{H}$ with vanishing trace; $S_0^2(\mathcal{H})$ is a Euclidean vector space with respect to the natural scalar product $\langle C, C'\rangle = \operatorname{trace}(CC')$, $C, C' \in S_0^2(\mathcal{H})$. We define the (reduced) *moduli space* [7] as

$$\mathcal{K}_0 = \mathcal{K}_0(\mathcal{H}) = \{C \in S_0^2(\mathcal{H}) \mid C + I \geq 0\},$$

where $\geq$ means positive semidefinite.

We observe that $\mathcal{K}_0$ is a convex body in $S_0^2(\mathcal{H})$. The interior of $\mathcal{K}_0$ consists of those $C \in \mathcal{K}_0$ for which $C + I > 0$, and the boundary of $\mathcal{K}_0$ consists of those $C \in \mathcal{K}_0$ for which $C + I$ has nontrivial kernel. The eigenvalues of the elements in $\mathcal{K}_0$ are contained in $[-1, \dim \mathcal{H} - 1]$. Hence $\mathcal{K}_0$ is compact. Finally, an easy argument using $\mathrm{GL}(\mathcal{H})$-invariance of $\mathcal{K}_0$ shows that the centroid of $\mathcal{K}_0$ is the origin.

Let M be a compact Riemannian manifold and $\mathcal{H} = \mathcal{H}_\lambda$ the eigenspace of the Laplacian $\triangle^M$ (acting on functions of M) corresponding to an eigenvalue λ. The DoCarmo–Wallach moduli space that parameterizes spherical minimal immersions $f : M \to S_V$ of M into the unit sphere S_V of a Euclidean vector space V, for various V, is the intersection $\mathcal{K}_0 \cap \mathcal{E}_\lambda$, where $\mathcal{E}_\lambda$ is a linear subspace of $S_0^2(\mathcal{H}_\lambda)$. Here f is an isometric minimal immersion of $\dim M/\lambda$ times the original metric of M. (For further details, see [3; 6; 8].) Intersecting $\mathcal{K}_0$ further with suitable linear subspaces of $\mathcal{E}_\lambda$, we obtain moduli that parameterize spherical minimal immersions with additional geometric properties (such as higher-order isotropy, equivariance with respect to an acting group of isometries of M, etc.).

A result of Moore [4] states that a spherical minimal immersion $f : S^m \to S^n$ with $n \leq 2m - 1$ is totally geodesic; in particular, the image of f is a great m-sphere in S^n. An important example showing that the upper bound is sharp is provided by the *tetrahedral minimal immersion* $f : S^3 \to S^6$ (see [2; 6]). Here f is $\mathrm{SU}(2)$-equivariant and non–totally geodesic. The name comes from the fact that the invariance group of f is the binary tetrahedral group $\mathbf{T}^* \subset S^3 = \mathrm{SU}(2)$, so that f factors through the canonical projection $S^3 \to S^3/\mathbf{T}^*$ and gives a minimal *imbedding* $\bar{f} : S^3/\mathbf{T}^* \to S^6$ of the tetrahedral manifold $S^3/\mathbf{T}^*$ into S^6.

Let $M = S^3$ and let $\mathcal{H}_{\lambda_p}$ be the pth eigenspace of the Laplacian on S^3 corresponding to the eigenvalue $\lambda_p = p(p+2)$. According to a result in [5; 6] there

Received January 28, 2003. Revision received April 15, 2003.