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Completions of Normal Affine Surfaces with a Trivial Makar-Limanov Invariant

Adrien Dubouloz

Introduction

For a connected normal affine surface $V = \mathrm{Spec}(A)$ over $\mathbb{C}$, the Makar-Limanov invariant of V [10] is the subalgebra $\mathrm{ML}(V) \subset A$ of all regular functions invariant under every algebraic $\mathbb{C}_+$-action on V. Constant functions are certainly contained in $\mathrm{ML}(V)$, and we say that the Makar-Limanov invariant of V is *trivial* (or that V is an *ML-surface*) if $\mathrm{ML}(V) = \mathbb{C}$. In [1], Bandman and Makar-Limanov have re-discovered a link between nonsingular ML-surfaces and *geometrically quasihomogeneous* surfaces studied by Gizatullin in [6]—that is, surfaces whose automorphism group has a Zariski open orbit with a finite complement. More precisely, they have established that, on a nonsingular ML-surface V, there exist at least two nontrivial algebraic $\mathbb{C}_+$-actions that generate a subgroup H of the automorphism group $\mathrm{Aut}(V)$ of V such that the orbit $H.v$ of a general closed point $v \in V$ has finite complement. By Gizatullin [6], such a surface is rational and is either isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ or can be obtained from a nonsingular projective surface $\bar{V}$ by deleting an ample divisor of a special form, called a *zigzag*. This is just a linear chain of nonsingular rational curves. Conversely, a nonsingular surface V completable by a zigzag is rational and geometrically quasihomogeneous (see [6]). In addition, if V is not isomorphic to $\mathbb{C}^* \times \mathbb{A}^1$ then it admits two independent $\mathbb{C}_+$-actions. More precisely, Bertin [2] showed that if V admits a $\mathbb{C}_+$-action then this action is unique unless V is completable by a zigzag. Altogether, this leads to the following result.

Theorem [1; 2; 6]. *A nonsingular affine surface V that is nonisomorphic to $\mathbb{C}^* \times \mathbb{A}^1$ has a trivial Makar-Limanov invariant if and only if V is completable by a zigzag.*

More generally, in this paper we prove the following theorem.

Theorem. *A normal affine surface V that is nonisomorphic to $\mathbb{C}^* \times \mathbb{A}^1$ has a trivial Makar-Limanov invariant if and only if V is completable by a zigzag.*

We are grateful to the referee for pointing out that closely related results are proved in two recent preprints [3; 14], under the additional assumption that V is rational.

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