Michigan Math. J. 52 (2004)

Rational Approximation on the Unit Sphere in $\mathbf{C}^2$

John T. Anderson, Alexander J. Izzo, & John Wermer

1. Introduction

For a compact set $X \subset \mathbf{C}^n$, we denote by $R(X)$ the closure in $C(X)$ of the set of rational functions holomorphic in a neighborhood of X. We are interested in finding conditions on X which imply that $R(X) = C(X)$, that is, conditions implying that each continuous function on X is the uniform limit of a sequence of rational functions holomorphic in a neighborhood of X.

When $n = 1$, the theory of rational approximation is well developed. Examples of sets without interior for which $R(X) \neq C(X)$ are well known, the "Swiss cheese" being a prime example. On the other hand, the Hartogs–Rosenthal theorem states that if the two-dimensional Lebesgue measure of X is zero, then $R(X) = C(X)$.

In higher dimensions, there is an obstruction to rational approximation that does not appear in the plane. For $X \subset \mathbf{C}^n$, we denote by $\hat{X}_r$ the rationally convex hull of X, which can be defined as the set of points $z \in \mathbf{C}^n$ such that every polynomial Q with $Q(z) = 0$ vanishes at some point of X. The condition $X = \hat{X}_r$ (X is rationally convex) is both necessary for rational approximation and difficult to establish, in practice, when $n > 1$; in the plane, every compact set is rationally convex.

We will consider primarily subsets of the unit sphere ∂B in $\mathbf{C}^2$. We have been motivated by a desire to obtain an analogue of the Hartogs–Rosenthal theorem in this setting. Basener [5] has given examples of rationally convex sets $X \subset \partial B$ for which $R(X) \neq C(X)$; his examples have the form $\{(z, w) \in \partial B : z \in E\}$, where $E \subset \mathbf{C}$ is a suitable Swiss cheese. These sets have the property that $\sigma(X) > 0$, where σ is three-dimensional Hausdorff measure on ∂B. It is reasonable to conjecture that if X is rationally convex and $\sigma(X) = 0$, then $R(X) = C(X)$. This paper contains several contributions to the study of this question.

In Section 2 we employ a construction of Henkin [10]. For a measure μ supported on ∂B orthogonal to polynomials, Henkin produced a function $K_\mu \in L^1(d\sigma)$ satisfying $\bar{\partial}_b K_\mu = -4\pi^2\mu$. Lee and Wermer established that, if $X \subset \partial B$ is rationally convex and if $\mu \in R(X)^\perp$ (i.e., $\int g\, d\mu = 0$ for all $g \in R(X)$), then K_μ extends holomorphically to the unit ball. We show that, if the extension belongs

Received October 7, 2002. Revision received December 19, 2002.