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A Proof of the Gap Labeling Conjecture

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1. Introduction

The "gap labeling conjecture" as formulated by Bellissard [3] is a statement—about the possible gaps in the spectrum of certain Schrödinger operators—that arises in solid state physics. It has a reduction to a purely mathematical statement about the range of the trace on a certain crossed-product C^*-algebra (see [13]). By a *Cantor set* we mean a compact, totally disconnected metric space without isolated points. A group action is *minimal* if every orbit is dense.

Theorem 1.1. *Let Σ be a Cantor set and let $\Sigma \times \mathbb{Z}^n \to \Sigma$ be a free and minimal action of $\mathbb{Z}^n$ on Σ with invariant probability measure μ. Let $\mu\colon C(\Sigma) \to \mathbb{C}$ and $\tau_\mu\colon C(\Sigma) \rtimes \mathbb{Z}^n \to \mathbb{C}$ be the traces induced by μ and denote likewise the induced maps on K-theory. Then*

$$\mu(K_0(C(\Sigma))) = \tau_\mu(K_0(C(\Sigma) \rtimes \mathbb{Z}^n)).$$

Note that $K_0(C(\Sigma))$ is isomorphic to $C(\Sigma, \mathbb{Z})$, the group of integer-valued continuous functions on Σ, and that the image under μ is the subgroup of $\mathbb{R}$ generated by the measures of the clopen subsets of Σ.

We will give a proof of this conjecture in this paper. It was also proved independently by Bellissard, Benedetti, and Gambaudo [2] and by Benameur and Oyono-Oyono [4].

The strategy of the proof is to use Connes's index theory for foliations but in the form presented in the book by Moore and Schochet [11]. In fact, this approach underlies all three proofs [2; 4]. Thus, one may apply the index theorem to "foliated spaces", which are more general than foliations. These are spaces that have a cover by compatible flow boxes as in the case of genuine foliations, except that the transverse direction is not required to be $\mathbb{R}^n$. In the case at hand it is a Cantor set.

There are two steps in the proof. The main one is to show that

$$\tau_\mu(K_0(C(\Sigma) \rtimes \mathbb{Z}^n)) \subseteq \mu(K_0(C(\Sigma))). \tag{1.1}$$

This will be carried out in Section 4. The reverse containment is easier and is proved in Section 2. The authors would like to thank Ryszard Nest for several interesting discussions on this material.

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