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Combinatorial Method in Adjoint Linear Systems on Toric Varieties

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1. Introduction

Inspired by minimal model theory, Fujita in the 1980s conjectured as follows.

Conjecture [6]. *Let X be a nonsingular complex projective variety of dimension n, and let D be an ample divisor on X. Then*:

(I) *$K_X + \ell D$ is generated by global sections for $\ell \geq n + 1$; and*
(II) *$K_X + \ell D$ is very ample for $\ell \geq n + 2$.*

Moreover, (I) *and* (II) *should still hold true if X has only "mild singularities".*

For nonsingular varieties, the one-dimensional case is an easy fact in curve theory. The two-dimensional case follows from the work of Reider [16]. In higher-dimensional cases, (I) is known for $n = 3$ [3] and $n = 4$ [8], and by [1] we know that $K_X + \frac{1}{2}(n^2 + n + 2)D$ is generated by global sections for all n. Less is known about (II) with one exception: if D is already very ample, then (I) and (II) follow from Bertini's theorem by induction on dimensions.

For part (I), allowing X to have rational Gorenstein singularities, Fujita himself had shown (among other things) that $K_X + (n + 1)D$ is nef. For varieties over a field of arbitrary characteristic that have singularities of F-rational type, Smith showed that (I) holds if D is further assumed to be generated by global sections [17]. (In characteristic zero this can also be proved by using vanishing theorems.) Both [6] and [17] apply well to quite general toric varieties, since they have only rational singularities and on them a Cartier divisor is nef if and only if it is basepoint free (cf. Section 5). Moreover, ample divisors are automatically generated by global sections (Corollary 2.3). In fact, for *nonsingular* toric varieties, Fujita's conjectures hold because ample divisors are automatically very ample (Demazure's theorem).

These implications motivate our present work: results on toric varieties should admit direct proofs using only toric (combinatorial) techniques. In this note such elementary proofs are found for rather general toric varieties. Moreover, our combinatorial treatment also provides results on the "very ampleness" conjecture (II).

Main Theorem. *Let X be a complete toric variety of dimension n with ample* (*Cartier*) *divisor D.*

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