Michigan Math. J. 51 (2003)

Möbius Transformations, the Carathéodory Metric, and the Objects of Complex Analysis and Potential Theory in Multiply Connected Domains

Steven R. Bell

1. Introduction

Let f_b denote the Riemann mapping function associated to a point b in a simply connected planar domain $\Omega \neq \mathbb{C}$. Everyone knows that f_b is the solution to an extremal problem; it is the holomorphic map h of Ω into the unit disc such that $h'(b)$ is real and as large as possible. Everyone knows also that all the maps f_b can be expressed in terms of a single Riemann map f_a associated to a point $a \in \Omega$ via

$$f_b(z) = \lambda \frac{f_a(z) - f_a(b)}{1 - f_a(z)\overline{f_a(b)}}, \tag{1.1}$$

where the unimodular constant λ is given by

$$\lambda = \frac{\overline{f_a'(b)}}{|f_a'(b)|}.$$

In this paper, I shall prove that solutions to the analogous extremal problems on a finitely multiply connected domain in the plane, the Ahlfors mappings, can be expressed in terms of just *two* fixed Ahlfors mappings. Many similarities with formula (1.1) in the simply connected case will become apparent, and I will explore some of the algebraic objects that present themselves. A by-product of these considerations will be that the infinitesimal Carathéodory metric on a multiply connected domain is simply a rational combination of two Ahlfors maps times one of their derivatives. I will explain an outlook that reveals a natural way to view the extremal functions involved in the definition of the Carathéodory metric "off the diagonal" in such a way that they extend to $\hat{\Omega} \times \hat{\Omega}$, where $\hat{\Omega}$ is the double of Ω.

I will also investigate the complexity of the classical Green's function and Bergman kernel associated to a multiply connected domain. In particular, it is proved in Section 6 that if Ω is a finitely connected domain in the plane such that no boundary component is a point, then there exist two Ahlfors maps f_a and f_b associated to Ω such that the Bergman kernel for Ω is given by

$$K(w,z) = \frac{f_a'(w)\overline{f_a'(z)}}{(1 - f_a(w)\overline{f_a(z)})^2}\left(\sum_{j,k=1}^{N} \lambda_{jk} H_j(w)\overline{H_k(z)}\right),$$

Received January 16, 2002. Revision received August 20, 2002.
Research supported by NSF Grant no. DMS-0072197.