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On Parameter Spaces for Artin Level Algebras

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Let $R = k[x_1, \dots, x_n]$ denote a polynomial ring and let $\underline{h} : \mathbb{N} \to \mathbb{N}$ be a numerical function. Consider the set of all graded Artin level quotients $A = R/I$ having Hilbert function $\underline{h}$. This set (if nonempty) is naturally in bijection with the closed points of a quasiprojective scheme $\mathcal{L}^{\circ}(\underline{h})$. The object of this note is to prove some specific geometric properties of these schemes, especially for $n = 2$. The case of Gorenstein Hilbert functions (i.e., where A has type 1) has been extensively studied, and several qualitative and quantitative results are known (see [17]). Our results should be seen as generalizing some of them to the non-Gorenstein case.

After establishing notation, we summarize the results in the next section. See [12; 17] as general references for most of the constructions used here.

1. Notation and Preliminaries

The base field k will be algebraically closed and of characteristic 0 (but see Remark 4.11). Let V be an n-dimensional k-vector space, and let

$$R = \bigoplus_{i \ge 0} \operatorname{Sym}^i V^*, \qquad S = \bigoplus_{i \ge 0} \operatorname{Sym}^i V.$$

Let $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ be dual bases of V^* and V (respectively), leading to identifications $R = k[x_1, \dots, x_n]$ and $S = k[y_1, \dots, y_n]$. There are internal products (see [11, p. 476])

$$\operatorname{Sym}^j V^* \otimes \operatorname{Sym}^i V \to \operatorname{Sym}^{i-j} V, \qquad u \otimes F \to u \cdot F,$$

making S into a graded R-module. This action may be seen as partial differentiation; if $u(\underline{x}) \in R$ and $F(\underline{y}) \in S$, then

$$u \cdot F = u(\partial/\partial y_1, \dots, \partial/\partial y_n) F.$$

If $I \subseteq R$ is a homogeneous ideal, then I^{-1} is the R-submodule of S defined as $\{F \in S : u \cdot F = 0 \text{ for all } u \in I\}$. This module (called Macaulay's inverse system for I) inherits a grading from S, so $I^{-1} = \bigoplus_i (I^{-1})_i$. Reciprocally, if $M \subseteq S$ is a graded submodule, then $\operatorname{ann}(M) = \{u : u \cdot F = 0 \text{ for all } F \in M\}$ is a homogeneous ideal in R. In classical terminology, if $u \cdot F = 0$ and $\deg u \le \deg F$, then u and F are said to be *apolar* to each other.

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