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Total Masses of Mixed Monge–Ampère Currents

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1. Introduction

Our starting point is the classical problem on numeric characteristics for zero sets of polynomial mappings $P: \mathbf{C}^n \to \mathbf{C}^m$. If $m \ge n$ and P has discrete zeros then this is about the total number of zeros counted with multiplicities, and for $m < n$ the characteristics are the projective volumes of the corresponding holomorphic chains Z_P. When $m = 1$ the volume equals the degree of the polynomial P, but for $m > 1$ the situation becomes much more difficult. In particular, in the general case no exact formulas can be obtained in terms of the exponents and the problem reduces to finding appropriate upper bounds. An example of such a bound is given by Bezout's theorem: If $m = n$ and P has discrete zeros, then their number does not exceed the product of the degrees of the components of P. An alternative estimate is due to Kouchnirenko [11; 12]: The number of zeros is at most $n!$ times the volume of the *Newton polyhedron of P at infinity* (the convex hull of all exponents of P and the origin). A refined version of the latter result was obtained by Bernstein [3], who showed that the number of (discrete) zeros of a Laurent polynomial mapping P on $(\mathbf{C} \setminus \{0\})^n$ is not greater than $n!$ times the mixed volume of the *Newton polyhedra* (the convex hulls of the exponents) of the components of P.

Here we put this problem into a wider context of pluripotential theory. This can be done by considering plurisubharmonic functions $u = \log|P|$ and studying the Monge–Ampère operators $(dd^c u)^p$; we use the notation $d = \partial + \bar\partial$ and $d^c = (\partial - \bar\partial)/2\pi i$. The key relation is the King–Demailly formula, which implies that if the codimension of the zero set is at least p then $(dd^c u)^p \ge Z_P$ (with an equality if $p = m \le n$). The problem of estimating total masses of the Monge–Ampère operators of plurisubharmonic functions u of logarithmic growth was studied in [22]. In particular, a relation was obtained in terms of the volume of a certain convex set generated by the function u, which in case $u = \log|P|$ is just the Newton polyhedron of P at infinity.

On the other hand, we know that the holomorphic chain Z_P with $m = p \le n$ can be represented as the wedge product of the currents (divisors) $dd^c \log|P_k|$, $1 \le k \le m$, which leads to consideration of the *mixed* Monge–Ampère operators $dd^c u_1 \wedge \cdots \wedge dd^c u_m$ and estimating their total masses. Another motivation for this problem are generalized degrees $\int_{\mathbf{C}^n} T \wedge (dd^c\varphi)^p$ of positive closed currents T with respect to plurisubharmonic weights φ, due to Demailly [5].

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